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Index theorems on torsional geometries

Tetsuji Kimura

Yukawa Institute for Theoretical Physics, Kyoto University,
Sakyo-ku, Kyoto 606-8502, Japan
E-mail: tetsuji@yukawa.kyoto-u.ac.jp

Abstract: We study various topological invariants on a torsional geometry in the presence of a totally anti-symmetric torsion H under the closed condition $\mathrm{d}H = 0$, which appears in string theory compactification scenarios. By using the identification between the Clifford algebra on the geometry and the canonical quantization condition of fermions in quantum mechanics, we construct $\mathcal{N} = 1$ quantum mechanical sigma model in the Hamiltonian formalism. We extend this model to $\mathcal{N} = 2$ system, equipped with the totally anti-symmetric tensor associated with the torsion on the target space geometry. Next we construct transition elements in the Lagrangian path integral formalism and apply them to the analyses of the Witten indices in supersymmetric systems. We explicitly show the formulation of the Dirac index on the torsional manifold which has already been studied. We also formulate the Euler characteristic and the Hirzebruch signature on the torsional manifold.

Keywords: Anomalies in Field and String Theories, Superstrings and Heterotic Strings, Differential and Algebraic Geometry, Flux compactifications.

Contents

1. Introduction

Flux compactification scenarios have become one of the most significant issues in the study of low energy effective theories from string theories (for instance, see [1−3] and references therein). Non-trivial fluxes induce a superpotential, which stabilizes moduli of a compactified geometry and decreases the number of "redundant" massless modes in the low energy effective theory in four dimensional spacetime. This mechanism, called the moduli stabilization, also gives a new insight into cosmology as well as string phenomenology ([4] and a huge number of related works).

Flux compactification provides another interesting issue to the compactified geometry itself: In a specific situation, for instance, the NS-NS three-form flux H_{mnp} behaves as a torsion on the compactified geometry and gives rise to a significant modification [5], i.e., the Kähler form is no longer closed. This phenomenon indicates that the fluxes modify the background geometry in supergravity in a crucial way. Of course, the Calabi-Yau condition [6] should be influenced by the back reactions from the fluxes onto the geometry.

If a certain n-dimensional manifold has a non-trivial structure group G on its tangent bundle, this manifold, called the G-structure manifold, admits the existence of nowhere vanishing tensors; for example, the metric ($G \subseteq \mathrm{O}(n)$), the Levi-Civita anti-symmetric tensor ($G \subseteq \mathrm{SO}(n)$), the almost complex structure ($G \subseteq \mathrm{U}(m)$ where $n = 2m$), and the holomorphic m-form ($G \subseteq \mathrm{SU}(m)$). This classification does not exclude the existence of torsion. (In this sense, a Calabi-Yau n-fold is one of the $\mathrm{SU}(n)$-structure manifolds.) This classification is also studied in terms of Killing spinors on the manifold. In particular, the six-dimensional SU(3)-structure manifold has been investigated in terms of intrinsic torsion [7] and has been applied to the string theory compactification scenarios [8]. Since we mainly study supergravity theories as low energy effective theories of string theories, we always assume the existence of the metric g_{mn} and dilaton field Φ on the compactified manifold. In a generic case of the string compactification, we can also introduce non-trivial NS-NS three-form flux H_{mnp} with its Bianchi identity. In type II theories appropriate R-R fluxes are also incorporated. All of these are strongly related via the preserved condition of supersymmetry. In the heterotic case, supersymmetry variations of the gravitino ψ_m, the dilatino λ and the gaugino χ give rise to the Killing spinor equations

$$0 = \delta\psi_m = \Big(\partial_m + \frac{1}{4}\omega_{-m}{}^{ab}\,\Gamma_{ab}\Big)\eta_+ \equiv D_m(\omega_-)\eta_+ \, , \tag{1.1a}$$

$$0 = \delta\lambda = -\frac{1}{4}\Big(\Gamma^m\nabla_m\Phi - \frac{1}{6}H_{mnp}\Gamma^{mnp}\Big)\eta_+ \, , \tag{1.1b}$$

$$0 = \delta\chi = -\frac{1}{4}F_{mn}\Gamma^{mn}\eta_+ \, , \tag{1.1c}$$

where η_+ is the Weyl spinor on the six-dimensional manifold whose normalization is given as $\eta_+^\dagger\eta_+ = 1$, and $\omega_{-mab} = \omega_{mab} - H_{mab}$ [5]. Then the NS-NS three-form flux H_{mnp} is interpreted as a totally anti-symmetric contorsion (or equivalently, a totally anti-symmetric torsion) on the manifold with negative sign: $H^m{}_{np} = -T^m{}_{np} = -\Gamma^m{}_{[np]}$. The analysis of the manifold becomes much clear when we introduce a set of mathematical definitions such as

$$\text{Almost complex structure}: \quad J_m{}^n \equiv i\eta_+^\dagger \Gamma_m{}^n \eta_+ \, , \quad J_m{}^p J_p{}^n = -\delta_m^n \, , \tag{1.2a}$$

$$\text{Lee-form}: \quad \theta \equiv J \lrcorner \mathrm{d}J = \frac{3}{2} J^{mn} \nabla_{[m} J_{np]} \, \mathrm{d}x^p \, , \tag{1.2b}$$

$$\text{Nijenhuis tensor}: \quad N_{mn}{}^p \equiv J_m{}^q \nabla_{[q} J_{n]}{}^p - J_n{}^q \nabla_{[q} J_{m]}{}^p \, , \tag{1.2c}$$

$$\text{Bismut torsion}: \quad T^{(\mathrm{B})}_{mnp} \equiv \frac{3}{2} J_m{}^q J_n{}^r J_p{}^s \nabla_{[s} J_{qr]} = -\frac{3}{2} J_{[m}{}^q \nabla_{|q|} J_{np]} \, . \tag{1.2d}$$

If there are no fermion condensations and H-flux condensation in heterotic string compactified on the manifold with SU(3)-structure satisfying $D_m(\omega_-) J_{np} = 0$, the compactified manifold is complex and non-Kähler. Actually this is so-called a conformally balanced manifold, on which the Nijenhuis tensor vanishes $\mathcal{N}_{mn}{}^p = 0$, the dilaton field is related to the Lee-form $\theta = 2\mathrm{d}\Phi$ and $\mathrm{d}(\mathrm{e}^{-2\Phi} J \wedge J) = 0$. Furthermore, the NS-NS three-form flux H_{mnp} is given by the Bismut torsion $T^{(\mathrm{B})}_{mnp}$ [9]. We can classify compactified manifolds under specific conditions in the following way (see also the discussions in [10 − 12]):

$$\theta = 2\mathrm{d}\Phi \, , \quad \mathrm{d}\big(\mathrm{e}^{-2\Phi} J \wedge J\big) = 0 \qquad \to \text{conformally balanced} \tag{1.3a}$$

$$\text{if } \theta = 0 \qquad \to \text{balanced} \tag{1.3b}$$

$$\text{if } \mathrm{d}\big(\mathrm{e}^{-\Phi} J\big) = 0 \qquad \to \text{conformally Kähler} \tag{1.3c}$$

$$\text{if } \mathrm{d}H = \mathrm{d}T^{(\mathrm{B})} = 0 \qquad \to \text{strong Kähler with torsion} \tag{1.3d}$$

On the contrary, however, one has not understood a lot of mathematical properties of the G-structure manifold such as moduli and moduli spaces. This is quite different from the case of Calabi-Yau manifold [13]. Because of the lack of knowledge, one has not been able to discuss the massless modes on the ground state in the effective theory derived from string theory compactified on the G-structure manifold.

Similarly, various kinds of topological invariants on torsional geometries have not been analyzed, although many topological invariants on Riemannian manifolds have been well investigated. Here let us briefly introduce some invariants: Suppose there exist Dirac fermions in an even dimensional geometry. We define chirality on the Dirac fermions and find the difference between the number of fermions with positive chirality and the number of fermions with negative chirality at the massless level. This difference is a topological invariant, which is called the index of the Dirac operator, or the Dirac index [14 − 16]. We also introduce the Euler characteristic as the difference between the number of harmonic even-forms and the number of odd-forms on the manifold, and the Hirzebruch signature as the difference between the number of self-dual forms and the number of anti-self-dual forms. These invariants are described in terms of polynomials of Riemann curvature two-form (see, for example, [17 − 19]). So far the index of the Dirac operator in the presence of torsion has been studied [20 − 23]. Unfortunately, however, the other indices on a torsional manifold have not been analyzed so much. In particular, it is quite worth studying the

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Euler characteristic on a complex manifold in the presence of torsion, which will give a new insight on the number of generation in the flux compactification scenarios.

The main discussion of this paper is to analyze such kinds of topological invariants derived from the Dirac operator, which appears in the following equations of motion for fermionic fields in the supergravity [24]:

$$0 = \not{D}(\omega)\lambda - \frac{1}{12} H_{mnp} \Gamma^{mnp} \lambda = \not{D}\Big(\omega - \frac{1}{3} H\Big)\lambda \, , \tag{1.4a}$$

$$0 = \not{D}(\omega, A)\chi - \frac{1}{12} H_{mnp} \Gamma^{mnp} \chi = \not{D}\Big(\omega - \frac{1}{3} H, A\Big)\chi \, . \tag{1.4b}$$

First, we define the index of the Dirac operator on the torsional manifold in the infinity limit of β:

$$\mathrm{index} \not{D} \equiv \lim_{\beta \to \infty} \mathrm{Tr}\big\{\Gamma_{(5)} \mathrm{e}^{-\beta \mathscr{R}}\big\} = \lim_{\beta \to 0} \mathrm{Tr}\big\{\Gamma_{(5)} \mathrm{e}^{-\beta \mathscr{R}}\big\} \, , \tag{1.5}$$

where $\mathscr{R}$ is an appropriate regulator, given by the square of the Dirac operator (or, equivalently, the Laplacian) in a usual case. Notice that since a topological value is definitely independent of the continuous parameter β, we can take the zero limit $\beta \to 0$. This topological invariant can be represented as an appropriate quantum number in supersymmetric quantum mechanics [14] via the identification of the cohomology on the manifold with the supersymmetric states in the quantum mechanics. To investigate this, we define the Witten index in the quantum mechanics

$$\lim_{\beta \to 0} \mathrm{Tr}\big\{(-1)^F \mathrm{e}^{-\frac{\beta}{\hbar}\mathscr{H}}\big\} = \lim_{\beta \to 0} \int \mathrm{d}X \, \langle \, X \, | (-1)^F \mathrm{e}^{-\frac{\beta}{\hbar}\mathscr{H}} | \, X \, \rangle \, . \tag{1.6}$$

We identify (1.5) with (1.6) via the identification of the the regulator $\mathscr{R}$ and the chirality operator $\Gamma_{(5)}$ on the manifold with the Hamiltonian $\mathscr{H}$ and the fermion number operator $(-1)^F$ in the quantum mechanics, respectively. The trace Tr denotes the sum of all transition elements whose final states $\langle \, X \, |$ correspond to the initial states $| \, X \, \rangle$. Second, we rewrite the Witten index from the Hamiltonian formalism, as described above, to the Lagrangian path integral formalism. During this process, we introduce discretized transition elements and adopt the Weyl-ordered form in order to avoid any ambiguous ordering of quantum operators. Then we integrate out momentum variables and obtain the transition elements described in the configuration space path integral. Third, we discuss the Feynman rule which defines free propagators and interaction terms in the supersymmetric systems. Finally, we evaluate the Witten indices in the quantum mechanical nonlinear sigma models in appropriate ways. This procedure is summarized in a clear way by de Boer, Peeters, Skenderis and van Nieuwenhuizen [25], and Bastianelli and van Nieuwenhuizen [26]. We will apply this technique to the analysis of index theorems on the torsional manifold. To simplify the discussion, we impose the closed condition $\mathrm{d}H = 0$ on the NS-NS three-form in the same way as [21, 20]. This indicates that we only focus on the index theorems on the strong Kähler with torsion (1.3d). Although this condition is too strong to find the suitable solution in the heterotic string compactification with non-trivial fluxes [27, 24], it

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is still of importance to analyze the manifold with such condition, which also appears in type II string theory compactifications.

This paper is organized as follows: In section 2 we construct $\mathcal{N} = 1$ and $\mathcal{N} = 2$ quantum supersymmetric Hamiltonians equipped with a non-vanishing totally anti-symmetric field H_{mnp}, which can be regarded as the torsion on the manifold considered. In section 3 we describe the transition elements in the Hamiltonian formalism and rewrite them to functional path integrals in the Lagrangian formalism. We also prepare bosonic and fermionic propagators in the quantum mechanics. This transition elements play significant roles in the evaluation of the Witten indices in next sections. In section 4 and 5 the Witten index in $\mathcal{N} = 1$ supersymmetric quantum mechanical nonlinear sigma model is analyzed. First we review the Witten index associated with the Dirac index on a usual Riemannian manifold without boundary. Next we generalize the index on the manifold in the presence of non-trivial torsion H. We obtain an explicit expression of the Pontrjagin class and of the Chern character on the torsional manifold. The Euler characteristic corresponding to the Witten index in $\mathcal{N} = 2$ supersymmetric system is discussed in section 6. This topological invariant is also discussed on the torsional manifold. In section 7 we also analyze the derivation of the Hirzebruch signature on the manifold with and without torsion from the $\mathcal{N} = 2$ supersymmetric quantum mechanics. We summarize this paper and discuss open problems and future works in section 8. We attach some appendices in the last few pages. In appendix A we list the convention of differential geometry which we adopt in this paper. In appendix B a number of useful formulae, which play important roles in the computation of Feynman graphs, are listed.

2. Supersymmetric quantum Hamiltonians

First of all, we prepare a bosonic operator x^m and its canonical conjugate momentum p_m in quantum mechanics, whose canonical quantization condition is defined as a commutation relation between them in such a way as $[x^m, p_n] = i\hbar\,\delta^m_n$. Since we consider a quantum mechanical nonlinear sigma model, we regard x^m as a coordinate on the target space of the sigma model, where its index runs $m = 1, \ldots, D$. Since the target space is curved, the differential representation of the canonical momentum operator is given as $g^{\frac{1}{4}} p_m g^{-\frac{1}{4}} = -i\hbar\,\partial_m$ equipped with the determinant of the target space metric $g = \det g_{mn}$. We also introduce a real fermionic operator ψ^a in the quantum mechanics, equipped with the local Lorentz index $a = 1, \ldots, D$. In the quantum mechanics of real fermions, we define the canonical quantization condition as an anti-commutation relation $\{\psi^a, \psi^b\} = \hbar\,\delta^{ab}$. Since, under the identification $\psi^a \equiv \sqrt{\frac{\hbar}{2}}\Gamma^a$, the structure of this quantization condition can be interpreted as the SO(D) Clifford algebra given by the anti-commutation relation between the Dirac gamma matrices $\{\Gamma^a, \Gamma^b\} = 2\delta^{ab}$ on the target geometry, we will investigate the Dirac index on this curved geometry in terms of the Witten index in the quantum mechanics. First let us discuss $\mathcal{N} = 1$ supersymmetry, and extend this to $\mathcal{N} = 2$ supersymmetry under a certain condition. We should choose $\mathcal{N} = 1$ or $\mathcal{N} = 2$ in the case when we want to study the index density for the Pontrjagin classes, or for the Euler characteristics, respectively [14].[1]

[1]Alvarez-Gaumé [14] and Mavromatos [20] refer the $\mathcal{N} = 2$ ($\mathcal{N} = 1$) model to $\mathcal{N} = 1$ ($\mathcal{N} = 1/2$)

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2.1 $\mathcal{N} = 1$ real supersymmetry

Now let us introduce the $\mathcal{N} = 1$ supersymmetry algebra with respect to a real fermionic charge Q^1:

$$\{Q^1, Q^1\} = 2\hbar\, \mathscr{H}^1 \,. \tag{2.1}$$

Note that $\mathscr{H}^1$ is the quantum Hamiltonian in $\mathcal{N} = 1$ system, where the superscript "1" indicates $\mathcal{N} = 1$. We will realize this algebra in terms of quantum operators x^m, p_m and ψ^a. It is useful to introduce a covariant momentum operator associated with a covariant derivative $D_m(\omega - \frac{1}{3}H)$ which appears in the equation of motion in the supergravity (1.4). The covariant momentum operator is

$$\pi_m^{(-1/3)} \equiv p_m - \frac{\hbar}{2}\Big(\omega_{mab} - \frac{1}{3}H_{mab}\Big)\Sigma^{ab} \,. \tag{2.2}$$

Later we sometimes use the description $\hat{\omega}_{mab} \equiv \omega_{mab} - \frac{1}{3}H_{mab}$. Since the Dirac operator acts on spinors on the geometry, the Lorentz generator Σ^{ab} is given in the spinor representation, which can be described in terms of the real fermions via the identification $\Gamma^a = \sqrt{\frac{2}{\hbar}}\psi^a$ such as

$$\Sigma^{ab} = \frac{i}{4}\big(\Gamma^a\Gamma^b - \Gamma^b\Gamma^a\big) = \frac{i}{2\hbar}\big(\psi^a\psi^b - \psi^b\psi^a\big) \equiv \frac{i}{\hbar}\psi^{ab} \,. \tag{2.3}$$

We should also define the action of the covariant momentum on the fermionic operator:

$$g^{\frac{1}{4}}[\pi_m^{(-1/3)}, \psi^a]g^{-\frac{1}{4}} = 0 \,, \quad g^{\frac{1}{4}}[\pi_m^{(-1/3)}, \psi^n]g^{-\frac{1}{4}} = i\hbar\, \Gamma^n_{(-1/3)pm}\, \psi^p = i\hbar\Big(\Gamma^n_{0pm} - \frac{1}{3}H^n{}_{pm}\Big)\psi^p \,, \tag{2.4}$$

where Γ^n_{0pm} is the Levi-Civita connection defined in appendix A. Actually, the above commutator is associated with the covariant derivative of the Dirac gamma matrix on the target geometry.

By using the covariant momentum $\pi_m^{(-1/3)}$, let us represent the supercharge Q_H^1 and the Hamiltonian $\mathscr{H}_H^1$ (where the subscript H denotes that the operator contains the torsion H) as follows:

$$Q_H^1 \equiv \psi^m g^{\frac{1}{4}}\pi_m^{(-1/3)}g^{-\frac{1}{4}} = \psi^m g^{\frac{1}{4}}\Big(p_m - \frac{i}{2}\Big(\omega_{mab} - \frac{1}{3}H_{mab}\Big)\psi^{ab}\Big)g^{-\frac{1}{4}} \,, \tag{2.5a}$$

$$\mathscr{H}_H^1 = \frac{1}{2}g^{-\frac{1}{4}}\pi_m^{(-1)}g^{mn}\sqrt{g}\,\pi_n^{(-1)}\, g^{-\frac{1}{4}} + \frac{\hbar^2}{8}R(\omega) - \frac{\hbar^2}{24}H_{mnp}H^{mnp} \,. \tag{2.5b}$$

Note that we used the closed condition $\mathrm{d}H = 0$. Since we used the complete square in $\mathscr{H}_H^1$, the magnitude of the torsion in the covariant momentum is changed to $\pi_m^{(-1)}$. This is consistent with the analysis of the Killing spinor equation in the heterotic theory [24]. We can also formulate the $\mathcal{N} = 1$ supersymmetric charges with introducing a (non-abelian)

supersymmetric quantum mechanics.

gauge fields on the target space:

$$Q^1_H = \psi^m g^{\frac14}\widetilde{\pi}^{(-1/3)}_m g^{-\frac14} \,, \quad \{Q^1_H, Q^1_H\} = 2\hbar\,\mathscr{H}^1_H \,, \tag{2.6a}$$

$$\begin{aligned}\mathscr{H}^1_H &= \frac12 g^{-\frac14}\widetilde{\pi}^{(-1)}_m g^{mn}\sqrt{g}\,\widetilde{\pi}^{(-1)}_n g^{-\frac14} + \frac{\hbar^2}{8}\Big[R(\omega) - \frac13 H_{mnp}H^{mnp}\Big] \\ &\quad - \frac12 F^{\alpha}_{mn}\psi^{mn}(\hat{c}^\dagger T_\alpha \hat{c}) \,,\end{aligned} \tag{2.6b}$$

$$\widetilde{\pi}^{(\alpha)}_m = p_m - \frac{i}{2}\Big(\omega_{mab} + \alpha H_{mab}\Big)\psi^{ab} - iA^{\alpha}_m(\hat{c}^\dagger T_\alpha \hat{c}) \,, \tag{2.6c}$$

where we used the anti-hermitian matrix T_α as a generator of the gauge symmetry group. We also introduced a complex ghost field $\hat{c}^i$ living in the quantum mechanics.

2.2 $\mathcal{N} = 2$ complex supersymmetry

Now we introduce two sets of real fermionic operators ψ^a_α $(\alpha = 1, 2)$ and perform the complexification of fermionic operators via linear combination

$$\varphi^a \equiv \frac{1}{\sqrt{2}}(\psi^a_1 + i\psi^a_2) \,, \quad \overline{\varphi}^a \equiv \frac{1}{\sqrt{2}}(\psi^a_1 - i\psi^a_2) \,. \tag{2.7a}$$

Note that we used the convention $\overline{\varphi}^a = (\varphi^a)^\dagger$. Then the canonical quantization condition is extended in such a way as

$$\{\varphi^a, \varphi^b\} = 0 \,, \quad \{\overline{\varphi}^a, \overline{\varphi}^b\} = 0 \,, \quad \{\varphi^a, \overline{\varphi}^b\} = \hbar\,\delta^{ab} \,. \tag{2.7b}$$

This is nothing but the $\mathrm{SO}(D,D)$ Clifford algebra. This complex fermion φ^a plays a central role in $\mathcal{N} = 2$ supersymmetry, while ψ^a consists of $\mathcal{N} = 1$ supersymmetry. Now let us construct the $\mathcal{N} = 2$ supersymmetric model. Let us define the commutation relations between the covariant momentum operator π_m and the complex fermions, which are given in terms of the affine connection Γ^p_{0mn} in the same analogy as in the $\mathcal{N} = 1$ system:

$$g^{\frac14}[\pi_m, \varphi^n]g^{-\frac14} = i\hbar\,\Gamma^n_{0pm}\varphi^p \,, \quad g^{\frac14}[\pi_m, \varphi_n]g^{-\frac14} = -i\hbar\,\Gamma^p_{0nm}\varphi_p \,. \tag{2.8}$$

The Lorentz generator coupled to the spin connection and the curvature tensor are expressed as

$$\Sigma^{ab} = \frac{i}{\hbar}\Big(\varphi^a\overline{\varphi}^b - \varphi^b\overline{\varphi}^a\Big) \,, \tag{2.9a}$$

$$g^{\frac14}[\pi_m, \pi_n]\,g^{-\frac14} = \frac{i\hbar^2}{2}R_{abmn}(\omega)\Sigma^{ab} = -\hbar\,R_{abmn}(\omega)\varphi^a\overline{\varphi}^b \,. \tag{2.9b}$$

Next, let us express $\mathcal{N} = 2$ supercharge Q and extend it as the supercharge equipped with the torsion given by three-form flux H. In the same way as the $\mathcal{N} = 1$ supercharge, we will identify the de Rham cohomology on the manifold with the $\mathcal{N} = 2$ supersymmetry algebra. In the case on the Riemannian manifold, we identify the exterior derivative d on the geometry with the $\mathcal{N} = 2$ supercharge $Q \equiv \varphi^m g^{\frac14}\pi_m g^{-\frac14}$, where π_m is the covariant momentum in the $\mathcal{N} = 2$ quantum mechanics defined as

$$\pi_m = p_m - \frac{\hbar}{2}\omega_{mab}\,\Sigma^{ab} = p_m - i\omega_{mab}\,\varphi^a\overline{\varphi}^b \,. \tag{2.10}$$

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Let us introduce the torsion on the geometry. Following the discussions [20, 28 − 31], we extend the exterior derivative d to d_H in such a way as

$$\mathrm{d}_H \equiv \mathrm{d} + H\wedge \, , \quad (\mathrm{d}_H)^2 = (\mathrm{d}H) \wedge \ . \tag{2.11}$$

This means that d_H is nilpotent up to the derivative $\mathrm{d}H$, i.e., this yields the equivariant cohomology. In this paper we always impose the vanishing condition $\mathrm{d}H = 0$. In addition, by using the Darboux theorem, we can identify the one-form with the holomorphic variable, while the adjoint of the one-form can be identified with the anti-holomorphic variable. Thus, we identify the exterior derivative d_H and its adjoint $\mathrm{d}_H^\dagger$ with appropriate operators in terms of complex fermions φ^m and $\overline{\varphi}^m$ in the quantum mechanics:

$$\mathrm{d}_H \quad \leftrightarrow \quad Q_H \equiv \varphi^m g^{\frac{1}{4}} \pi_m g^{-\frac{1}{4}} + \alpha i H_{mnp} \, \varphi^m \varphi^n \varphi^p \, , \tag{2.12a}$$

$$\mathrm{d}_H^\dagger \quad \leftrightarrow \quad \overline{Q}_H \equiv \overline{\varphi}^m g^{\frac{1}{4}} \pi_m g^{-\frac{1}{4}} + \overline{\alpha} i H_{mnp} \, \overline{\varphi}^m \overline{\varphi}^n \overline{\varphi}^p \, . \tag{2.12b}$$

We wish to interpret Q_H as the "supercharge", associated with the exterior derivative $\mathrm{d}_H \equiv \mathrm{d} + H\wedge$, while $\overline{Q}_H$ associated with $\mathrm{d}_H^\dagger$, i.e., the adjoint of the derivative d_H. Here we also introduced the scale factor α, which should be fixed compared with the $\mathcal{N} = 1$ supercharge. In order to fix the coefficient α, let us truncate the supercharge Q_H to the supercharge Q_H^1 in the $\mathcal{N} = 1$ supersymmetry (2.5) via the restriction $\psi_2^a = 0$ and $\psi_1^a \equiv \psi^a$:

$$Q_H \to \frac{1}{\sqrt{2}} \psi^m \Big\{ g^{\frac{1}{4}} p_m \, g^{-\frac{1}{4}} - \frac{i}{2} \big(\omega_{mab} - \alpha H_{mab} \big) \psi^a \psi^b \Big\} = \frac{1}{\sqrt{2}} Q_H^1 \, . \tag{2.13}$$

Since we have already known the $\mathcal{N} = 1$ supercharge Q_H^1, we can fix the coefficient

$$\alpha = \frac{1}{3} = \overline{\alpha} \, . \tag{2.14}$$

Due to the first Bianchi identity $R_{[mnp]q}(\omega) = 0$ and $D_{[d}(\omega) H_{cab]} = \frac{1}{4} (\mathrm{d}H)_{dcab} = 0$, we find that the supersymmetry algebra is given by

$$\{Q_H, Q_H\} = \frac{\hbar}{6} \, (\mathrm{d}H)_{abcd} \, \varphi^{abcd} = 0 \, , \quad \{\overline{Q}_H, \overline{Q}_H\} = \frac{\hbar}{6} \, (\mathrm{d}H)_{abcd} \, \overline{\varphi}^{abcd} = 0 \, , \tag{2.15a}$$

$$\{Q_H, \overline{Q}_H\} = 2\hbar \, \mathscr{H}_H \, , \quad [Q_H, \mathscr{H}_H] = - \frac{1}{2\hbar} [\overline{Q}_H, (Q_H)^2] = 0 \, . \tag{2.15b}$$

The vanishing condition of the last commutator guarantees the supersymmetric system, in which the energy levels of the bosonic and fermionic states are degenerated. Now we explicitly express the Hamiltonian $\mathscr{H}_H$ in terms of the complex fermions:

$$Q_H = \varphi^m \Big(g^{\frac{1}{4}} p_m \, g^{-\frac{1}{4}} - i \omega_{mab} \, \varphi^a \overline{\varphi}^b + \frac{i}{3} H_{mab} \, \varphi^{ab} \Big) = \varphi^m \Big(g^{\frac{1}{4}} \pi_m \, g^{-\frac{1}{4}} + \frac{i}{3} H_{mab} \, \varphi^{ab} \Big) \, , \tag{2.16a}$$

$$\begin{aligned} \mathscr{H}_H = \; & \frac{1}{2} g^{-\frac{1}{4}} \Big\{ \pi_m + \frac{i}{2} H_{mab} \big(\varphi^{ab} + \overline{\varphi}^{ab} \big) \Big\} g^{mn} \sqrt{g} \Big\{ \pi_n + \frac{i}{2} H_{ncd} \big(\varphi^{cd} + \overline{\varphi}^{cd} \big) \Big\} g^{-\frac{1}{4}} \\ & - \frac{1}{2} R_{abmn}(\omega) \varphi^m \overline{\varphi}^n \varphi^a \overline{\varphi}^b + \frac{1}{6} \partial_m (H_{npq}) \Big(\overline{\varphi}^m \varphi^{npq} + \varphi^m \overline{\varphi}^{npq} - \frac{3\hbar}{2} g^{mn} \varphi^{pq} - \frac{3\hbar}{2} g^{mn} \overline{\varphi}^{pq} \Big) \\ & + \frac{1}{8} H_{mnr} H_{pq}{}^r \Big(\varphi^{mnpq} + \overline{\varphi}^{mnpq} - 2 \varphi^{mn} \overline{\varphi}^{pq} \Big) - \frac{\hbar}{2} H_{mpq} H_n{}^{pq} \, \varphi^m \overline{\varphi}^n + \frac{\hbar^2}{12} H_{mnp} H^{mnp} \, . \end{aligned} \tag{2.16b}$$

There exists a comment on the Hamiltonians in the $\mathcal{N}=1$ and in the $\mathcal{N}=2$ systems. The $\mathcal{N}=1$ Hamiltonian cannot be obtained by truncation of the $\mathcal{N}=2$ Hamiltonian, because the truncation $\psi_2^a=0$ is no longer consistent at the quantum level since the anti-commutation relation $\{\psi_1^a+i\psi_2^a, \psi_1^b+i\psi_2^b\}$ becomes non-zero via the truncation. On the other hand, we need not use such anti-commutation relation when we reduce the $\mathcal{N}=2$ supercharge to the charge in the $\mathcal{N}=1$ system.

3. Path integral formalism from Hamiltonian formalism

In this section we will discuss a generic strategy to obtain the transition element $\langle\, x\,|\mathrm{e}^{-\frac{\beta}{\hbar}\mathscr{H}}|\, y\,\rangle$ which appears in (1.6). We will introduce a number of useful tools to investigate the quantum mechanical path integral, i.e., the complete sets of eigenstates, and the Weyl-ordered form. Next we will move to the concrete constructions of the transition elements in the $\mathcal{N}=1$ and in the $\mathcal{N}=2$ systems. In this paper we omit many technical details which can be seen in the works [25, 26]. We mainly follow the convention defined in [26]. Before going to the main discussion, for later convenience, let us take a rescaling on the fermionic operators which we introduced in the previous section:

$$\mathcal{N}=2 \text{ system:} \quad \varphi^a \to \sqrt{\hbar}\,\varphi^a\ , \tag{3.1a}$$

$$\mathcal{N}=1 \text{ system:} \quad \psi^a \to \sqrt{\hbar}\,\psi^a\ , \qquad \text{ghost fields:} \quad \varphi^{\mathrm{gh}} \to \sqrt{\hbar}\,\varphi^{\mathrm{gh}}\ . \tag{3.1b}$$

3.1 General discussion

In order to formulate the transition elements we should prepare a number of tools. Let $\widehat{x}^m$ and $\widehat{p}_m$ be the operators of the coordinate and the momentum, respectively, while x^m and p_m denote their eigenvalues.[2] According to [25, 26], let us introduce the complete set of the $\widehat{x}$-eigenfunctions and the complete set of the $\widehat{p}$-eigenfunctions

$$\int \mathrm{d}^D x\,|\,x\,\rangle\sqrt{g(x)}\langle\, x\,| \equiv 1 \equiv \int \mathrm{d}^D p\,|\,p\,\rangle\langle\, p\,|\ , \tag{3.2}$$

where $g(x)=\det g_{mn}(x)$. We also define the inner products and the plane wave such as

$$\langle\, x\,|\,y\,\rangle \equiv \frac{1}{\sqrt{g(x)}}\delta^D(x-y)\ , \quad \langle\, p\,|\,p'\,\rangle \equiv \delta^D(p-p')\ , \tag{3.3a}$$

$$\langle\, x\,|\,p\,\rangle \equiv \frac{1}{(2\pi\hbar)^{D/2}}\exp\Big(\frac{i}{\hbar}p\cdot x\Big)g^{-\frac{1}{4}}\ , \tag{3.3b}$$

where the plane wave is normalized to

$$\int \mathrm{d}^D p\, \exp\left(\frac{i}{\hbar}p\cdot(x-y)\right) = (2\pi\hbar)^{D/2}\delta^D(x-y)\ , \tag{3.3c}$$

which appears when we evaluate the transition elements with infinitesimal short period. In order to discuss the path integrals for Dirac fermion operators, let us also introduce a set of

[2] The symbol "$\widehat{\ }$" on an operator is omitted if there are no confusions.

coherent states for fermionic operators in terms of the operator $\widehat{\varphi}^a$ satisfying $\{\widehat{\varphi}^a, \widehat{\overline{\varphi}}^b\} = \delta^{ab}$, and a complex Grassmann odd variable η:

$$| \eta \rangle \equiv \mathrm{e}^{\widehat{\overline{\varphi}}^a \eta^a} | 0 \rangle \, , \quad \widehat{\varphi}^a | 0 \rangle = 0 \, , \quad \widehat{\varphi}^a | \eta \rangle = \eta^a | \eta \rangle \, , \tag{3.4a}$$

$$\langle \overline{\eta} | \equiv \langle 0 | \mathrm{e}^{\overline{\eta}^a \widehat{\varphi}^a} \, , \quad \langle 0 | \widehat{\overline{\varphi}}^a = 0 \, , \quad \langle \overline{\eta} | \widehat{\overline{\varphi}}^a = \langle \overline{\eta} | \overline{\eta}^a \, . \tag{3.4b}$$

The inner product of these coherent state is given by $\langle \overline{\eta} | \zeta \rangle = \mathrm{e}^{\overline{\eta}^a \zeta^a}$. In the same analogy as (3.2), we introduce a complete set of the Dirac fermion coherent states:

$$1 = \int \prod_{a=1}^{D} \mathrm{d}\overline{\eta}^a \mathrm{d}\eta^a \, | \eta \rangle \, \mathrm{e}^{-\overline{\eta}^a \eta^a} \langle \eta | \, , \tag{3.5a}$$

$$\prod_{a=1}^{D} \mathrm{d}\overline{\eta}^a \equiv \mathrm{d}\overline{\eta}^D \mathrm{d}\overline{\eta}^{D-1} \cdots \mathrm{d}\overline{\eta}^1 \, , \quad \prod_{a=1}^{D} \mathrm{d}\eta^a \equiv \mathrm{d}\eta^1 \mathrm{d}\eta^2 \cdots \mathrm{d}\eta^D \, . \tag{3.5b}$$

Generically we define the following matrix element $M(z,y)$ in the quantum mechanics:

$$M(z,y) = \langle z | \widehat{\mathcal{O}}(\widehat{x}, \widehat{p}) | y \rangle \, , \tag{3.6}$$

where $| y \rangle$ and $\langle z |$ are the initial and final state, respectively. Now we are quite interested in the transition element with respect to the quantum Hamiltonian $\widehat{\mathscr{H}}$ and a parameter β:

$$T(z, \overline{\eta}; y, \zeta; \beta) \equiv \langle z, \overline{\eta} | \exp\Big(- \frac{\beta}{\hbar} \widehat{\mathscr{H}} \Big) | y, \zeta \rangle \, . \tag{3.7}$$

Next we introduce $N-1$ complete sets of position eigenstates x_k and of the fermion coherent states λ_k into the above transition elements. At the same time let us also insert N complete sets of momentum eigenstates p_k and of another fermion coherent states ξ_k to yield

$$\begin{aligned}
&\langle z, \overline{\eta} | \exp\Big(- \frac{\beta}{\hbar} \widehat{\mathscr{H}} \Big) | y, \zeta \rangle \\
&= \int \prod_{i=1}^{N-1} \mathrm{d}^D x_i \prod_{i'=1}^{N-1} \mathrm{d}\overline{\lambda}_{i'} \lambda_{i'} \, \mathrm{e}^{-\overline{\lambda}_{i'} \lambda_{i'}} \prod_{j=1}^{N} \mathrm{d}^D p_j \prod_{j'=0}^{N-1} \mathrm{d}\overline{\xi}_{j'} \mathrm{d}\xi_{j'} \, \mathrm{e}^{-\overline{\xi}_{j'} \xi_{j'}} \\
&\quad \times \prod_{k=0}^{N-1} \langle x_{k+1}, \overline{\lambda}_{k+1} | p_k, \xi_k \rangle \exp\bigg(- \frac{\epsilon}{\hbar} \mathscr{H}_{\mathrm{W}} \Big(x_{k+\frac{1}{2}}, p_{k+1}; \overline{\xi}_k, \frac{1}{2}(\xi_k + \lambda_k) \Big) \bigg) \langle p_k, \overline{\xi}_k | x_k, \lambda_k \rangle \\
&= \big[g(z) g(y) \big]^{-\frac{1}{4}} \int \prod_{j=1}^{N} \frac{\mathrm{d}^D p_j}{(2\pi\hbar)^D} \prod_{i=1}^{N-1} \mathrm{d}^D x_i \prod_{j'=0}^{N-1} \mathrm{d}\overline{\xi}_{j'} \mathrm{d}\xi_{j'} \\
&\quad \times \exp\left(\overline{\eta} \cdot \xi_{N-1} + \frac{\epsilon}{\hbar} \sum_{k=0}^{N-1} \Big[i p_{k+1} \cdot \frac{x_{k+1} - x_k}{\epsilon} - \hbar \overline{\xi}_k \cdot \frac{\xi_k - \xi_{k-1}}{\epsilon} - \mathscr{H}_W (x_{k+\frac{1}{2}}, p_{k+1}; \overline{\xi}_k, \xi_{k-\frac{1}{2}}) \Big] \right) .
\end{aligned} \tag{3.8}$$

Notice that the subscript k denotes the k-th complete set of the bosonic eigenstates, or the k-th complete set of the fermionic coherent states. We also note that $y = x_0$, $z = x_N$, $\overline{\eta} = \overline{\lambda}_N$, $\zeta = \lambda_0 = \xi_{-1}$. We adopt the midpoint rule $x_{k+\frac{1}{2}} = \frac{1}{2}(x_{k+1} + x_k)$ and $\xi_{k-\frac{1}{2}} =$

$\frac{1}{2}(\xi_k + \xi_{k-1})$. The factors $\sqrt{g(x_k)}$ compensate exactly the $g^{\frac{1}{4}}$ factors from the plane waves in the inner products. Furthermore, we integrated the arguments λ_k and $\overline{\lambda}_k$ to yield a useful equation

$$\int \mathrm{d}\overline{\lambda}_k \mathrm{d}\lambda_k \, \mathrm{e}^{-\overline{\lambda}_k \cdot (\lambda_k - \xi_{k-1})} f(\lambda_k) = f(\xi_{k-1}) \, , \tag{3.9}$$

where $f(\lambda)$ is an arbitrary function of the fermionic variable λ. Notice that $\widehat{\mathscr{H}}$ is the quantum Hamiltonian in terms of quantum operators, while $\mathscr{H}_{\mathrm{W}}$ is its Weyl-ordered form. The translation from the operator to the Weyl-ordered form is given in terms of the symmetrized form $\widehat{\mathscr{H}}_{\mathrm{S}}$ by

$$\widehat{\mathscr{H}} = \widehat{\mathscr{H}}_{\mathrm{S}} + \text{further terms} = \mathscr{H}_{\mathrm{W}} \, . \tag{3.10}$$

Integrating out the (discretized) momenta and taking the continuum limit $N \to \infty$, $\epsilon/\beta \to \mathrm{d}\tau$ with $\sum_{k=0}^{N-1} \epsilon/\beta \to \int_{-1}^{0} \mathrm{d}\tau$, we obtain the continuum path integral description in a following form:

$$T(z, \overline{\eta}; y, \zeta; \beta) = \left(\frac{g(z)}{g(y)} \right)^{\frac{1}{4}} \frac{1}{(2\pi\beta\hbar)^{D/2}} \, \mathrm{e}^{\overline{\eta}_a \zeta^a} \left\langle \exp \left(- \frac{1}{\hbar} S^{(\mathrm{int})} - \frac{1}{\hbar} S^{(\mathrm{source})} \right) \right\rangle_0 . \tag{3.11}$$

Note the followings: The action $S^{(\mathrm{int})}$ is given in terms of the interaction terms in the Lagrangian derived from the Legendre transformation of Weyl-ordered Hamiltonian, which we will explicitly show later. We introduced the external source of fields contained in the action $S^{(\mathrm{source})}$ to define their propagators. The additional factor $\sqrt{g(z)}$ appears due to the expanding the metric in $S^{(\mathrm{int})}$ at the point z and due to the integrating out the free kinetic terms of fields (see, for detail, section 2.1 in [26]). The symbol $\langle \cdots \rangle_0$ denotes the contraction of interaction terms in terms of propagators and setting the external source to zero. From now on we simply abbreviate $\big\langle \mathrm{e}^{-\frac{1}{\hbar} S^{(\mathrm{int})} - \frac{1}{\hbar} S^{(\mathrm{source})}} \big\rangle_0$ as $\big\langle \exp(-\frac{1}{\hbar} S^{(\mathrm{int})}) \big\rangle$.

3.2 Weyl-ordered form of quantum Hamiltonians

The next task is to study the Weyl-ordered form of the Hamiltonians $\mathscr{H}_H^{\mathrm{W}}$ and obtain the actions $S^{(\mathrm{int})}$ in the $\mathcal{N}=2$ and the $\mathcal{N}=1$ systems, respectively. The symmetrized form of the bosonic operators is defined by

$$\prod_{m,n} N! \big\{ (\widehat{p}_m)^{k_m} (\widehat{x}^n)^{\ell_n} \big\}_{\mathrm{S}} \equiv \prod_{m,n} \Big(\frac{\partial}{\partial \alpha^m} \Big)^{k_m} \Big(\frac{\partial}{\partial \beta_n} \Big)^{\ell_n} \big(\alpha^m \widehat{p}_m + \beta_n \widehat{x}^n \big)^N \, , \tag{3.12a}$$

$$N \equiv \sum_m k_m + \sum_n \ell_n \, . \tag{3.12b}$$

In the $\mathcal{N}=2$ complex fermions' case we define the following anti-symmetrized form:

$$\prod_{a,b} N! \big\{ (\widehat{\varphi}^a)^{m_a} (\widehat{\overline{\varphi}}_b)^{n_b} \big\}_{\mathrm{S}} \equiv \prod_{a,b} \Big(\frac{\partial}{\partial \alpha_a} \Big)^{m_a} \Big(\frac{\partial}{\partial \beta^b} \Big)^{n_b} \big(\alpha_a \widehat{\varphi}^a + \beta^b \widehat{\overline{\varphi}}_b \big)^N \, , \tag{3.13a}$$

$$N \equiv \sum_a m_a + \sum_b n_b \, , \tag{3.13b}$$

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where we perform the left derivative with respect to the Grassmann odd variables α_a and β^b. In the $\mathcal{N}=1$ real fermions' case, the anti-symmetrized form is defined by

$$\big(\psi^{a_1}\cdots\psi^{a_N}\big)_{\rm S} \equiv \frac{1}{N!}\prod_i \Big(\frac{\partial}{\partial\alpha_{a_i}}\Big)\big(\alpha_a\psi^a\big)^N . \tag{3.14}$$

By using the above rules, we obtain the Weyl-ordered form of the $\mathcal{N}=2$ Hamiltonian

$$\begin{aligned}\mathscr{H}_H^{\rm W} = {} & \frac{1}{2}\Big(g^{mn}\pi_m^{(-1)}\pi_n^{(-1)}\Big)_{\rm S} + \frac{\hbar^2}{8}\Big[g^{mn}\Gamma^p_{0mq}\Gamma^q_{0np} + g^{mn}\,\omega_{mab}\,\omega_n{}^{ab}\Big] \\ & - \frac{\hbar^2}{2}R_{pqmn}(\Gamma_0)(\varphi^m\overline{\varphi}^n\varphi^p\overline{\varphi}^q)_{\rm S} + \frac{\hbar^2}{12}H_{mnp}H^{mnp} + \frac{\hbar^2}{6}\partial_m(H_{npq})\Big[(\overline{\varphi}^m\varphi^{npq})_{\rm S} + (\varphi^m\overline{\varphi}^{npq})_{\rm S}\Big] \\ & + \frac{\hbar^2}{8}H_{mnr}H_{pq}{}^r\Big[(\varphi^{mnpq})_{\rm S} + (\overline{\varphi}^{mnpq})_{\rm S} - 2(\varphi^{mn}\overline{\varphi}^{pq})_{\rm S}\Big] ,\end{aligned} \tag{3.15a}$$

$$\pi_m^{(-1)} \equiv \pi_m + \frac{i\hbar}{2}H_{mab}\big(\varphi^{ab} + \overline{\varphi}^{ab}\big) = p_m - i\hbar\,\omega_{mab}\,\varphi^a\overline{\varphi}^b + \frac{i\hbar}{2}H_{mab}\big(\varphi^{ab}+\overline{\varphi}^{ab}\big) , \tag{3.15b}$$

and of the $\mathcal{N}=1$ Hamiltonian

$$\begin{aligned}\mathscr{H}_H^{1;\rm W} = {} & \frac{1}{2}\Big(g^{mn}\widetilde{\pi}_m^{(-1)}\widetilde{\pi}_n^{(-1)}\Big)_{\rm S} + \frac{\hbar^2}{8}\Big\{g^{mn}\Gamma^p_{0mq}\Gamma^q_{0np} + \frac{1}{2}g^{mn}\omega_{-mab}\,\omega_{-n}{}^{ab}\Big\} \\ & - \frac{\hbar^2}{24}H_{mnp}H^{mnp} - \frac{\hbar^2}{2}F^\alpha_{mn}(\psi^{mn})_{\rm S}(\hat{c}^\dagger T_\alpha\hat{c}) ,\end{aligned} \tag{3.16a}$$

$$\widetilde{\pi}_m^{(-1)} \equiv p_m - \frac{i\hbar}{2}\omega_{-mab}\,\psi^{ab} - i\hbar\,A^\alpha_m(\hat{c}^\dagger T_\alpha\hat{c}) . \tag{3.16b}$$

To proceed computations in path integral formalism in the $\mathcal{N}=1$ system, we would like to add a second set of "free" Majorana fermions in order to simplify the path integral in the $\mathcal{N}=1$ system in the same way as the one in the $\mathcal{N}=2$ system. Denoting the original Majorana fermions ψ^a by ψ_1^a, and the new ones by ψ_2^a, and combining them, we again construct Dirac fermions χ^a and $\overline{\chi}^a$ as

$$\chi^a = \frac{1}{\sqrt{2}}\big(\psi_1^a + i\psi_2^a\big) , \quad \overline{\chi}^a = \frac{1}{\sqrt{2}}\big(\psi_1^a - i\psi_2^a\big) . \tag{3.17}$$

Notice that, in this context, ψ_2^a differs from the second component of the previously defined Dirac fermions φ^a because now ψ_2^a is introduced as a "free" fermion in the $\mathcal{N}=1$ Hamiltonian.

3.3 Explicit form of the transition element in $\mathcal{N}=2$ system

We are ready to discuss the explicit form of the transition element in the $\mathcal{N}=2$ system in the framework of the Lagrangian formalism. Let us first decompose the bosonic and fermionic variables into two parts, i.e., the background fields and quantum fluctuations in such a way as $x^m(\tau) = x^m_{\rm bg}(\tau) + q^m(\tau)$ and $\xi^a(\tau) = \xi^a_{\rm bg}(\tau) + \xi^a_{\rm qu}(\tau)$, respectively. These background fields follow the free equations of motion whose solutions are

$$x^m_{\rm bg}(\tau) = z^m + \tau(z^m - y^m) , \quad \xi^a_{\rm bg}(\tau) = \zeta^a , \quad \overline{\xi}^a_{\rm bg}(\tau) = \overline{\eta}^a , \tag{3.18}$$

with constraints (via the mean-value theorem)

$$q^m(-1) = q^m(0) = 0 \; , \quad \int_{-1}^0 \mathrm{d}\tau \, q^m(\tau) = 0 \; , \tag{3.19a}$$

$$\xi^a_{\mathrm{qu}}(-1) = \overline{\xi}^a_{\mathrm{qu}}(0) = 0 \; . \tag{3.19b}$$

Then the description of the transition element in the configuration space path integral is given in the following form (see eq. (2.81) in [26]):

$$\begin{aligned}
\langle \, z, \overline{\eta} \, | \exp\Big(- \frac{\beta}{\hbar} \widehat{\mathscr{H}}_H \Big) | \, y, \zeta \, \rangle &= \Big(\frac{g(z)}{g(y)} \Big)^{\frac{1}{4}} \frac{1}{(2\pi\beta\hbar)^{D/2}} \, \mathrm{e}^{\overline{\eta}_a \zeta^a} \Big\langle \exp\Big(- \frac{1}{\hbar} S_H^{(\mathrm{int})} \Big) \Big\rangle , && (3.20\mathrm{a}) \\
\overline{\eta}_a \zeta^a - \frac{1}{\hbar} S_H^{(\mathrm{int})} &= - \frac{1}{\hbar} \big(S_H - S^{(0)} \big) , && (3.20\mathrm{b}) \\
- \frac{1}{\hbar} S_H &= - \frac{1}{\beta\hbar} \int_{-1}^0 \mathrm{d}\tau \, \frac{1}{2} g_{mn}(x) \Big(\frac{\mathrm{d}x^m}{\mathrm{d}\tau} \frac{\mathrm{d}x^n}{\mathrm{d}\tau} + b^m c^n + a^m a^n \Big) + \overline{\eta}_a \zeta^a \\
& \quad - \int_{-1}^0 \mathrm{d}\tau \, \delta_{ab} \, \overline{\xi}^a_{\mathrm{qu}} \frac{\mathrm{d}}{\mathrm{d}\tau} \xi^b_{\mathrm{qu}} \\
& \quad - \int_{-1}^0 \mathrm{d}\tau \, \frac{\mathrm{d}x^m}{\mathrm{d}\tau} \Big(\omega_{mab}(x) \overline{\xi}^a \xi^b - \frac{1}{2} H_{mab}(x) (\xi^{ab} + \overline{\xi}^{ab}) \Big) \\
& \quad + \frac{\beta\hbar}{2} \int_{-1}^0 \mathrm{d}\tau \, R_{cdab}(\omega(x)) \, \xi^a \overline{\xi}^b \xi^c \overline{\xi}^d \\
& \quad - \frac{\beta\hbar}{8} \int_{-1}^0 \mathrm{d}\tau \, H_{abe}(x) H_{cd}{}^e(x) \Big(\xi^{abcd} + \overline{\xi}^{abcd} - 2 \xi^{ab} \overline{\xi}^{cd} \Big) \\
& \quad - \frac{\beta\hbar}{6} \int_{-1}^0 \mathrm{d}\tau \, \partial_m (H_{npq}(x)) \Big(\overline{\xi}^m \xi^{npq} + \xi^m \overline{\xi}^{npq} \Big) \\
& \quad - \frac{\beta\hbar}{8} \int_{-1}^0 \mathrm{d}\tau \, \mathcal{G}_2(x) \; , && (3.20\mathrm{c}) \\
\mathcal{G}_2(x) &\equiv g^{mn}(x) \Big\{ \Gamma^p_{0mq}(x) \Gamma^q_{0np}(x) + \omega_{mab}(x) \omega_n{}^{ab}(x) \Big\} \\
& \quad + \frac{2}{3} H_{mnp}(x) H^{mnp}(x) \; , && (3.20\mathrm{d}) \\
- \frac{1}{\hbar} S^{(0)} &= - \frac{1}{\beta\hbar} \int_{-1}^0 \mathrm{d}\tau \, \frac{1}{2} g_{mn}(z) \Big(\frac{\mathrm{d}q^m}{\mathrm{d}\tau} \frac{\mathrm{d}q^n}{\mathrm{d}\tau} + b^m c^n + a^m a^n \Big) \\
& \quad - \int_{-1}^0 \mathrm{d}\tau \, \delta_{ab} \, \overline{\xi}^a_{\mathrm{qu}} \frac{\mathrm{d}}{\mathrm{d}\tau} \xi^b_{\mathrm{qu}} \; . && (3.20\mathrm{e})
\end{aligned}$$

Note that we introduced anti-commuting ghost fields b^m, c^m and a commuting ghost field a^m associated with the integrating out of momentum variables. They also appear in the $\mathcal{N} = 1$ system. We should notice that the metric in $S^{(0)}$ is given at the point z, not at the intermediate point x, while the metric, spin connection, and the fluxes in S_H are given at the intermediate point x. We can also define the propagators in this system:

$$\big\langle q^m(\sigma) q^n(\tau) \big\rangle = -\beta\hbar \, g^{mn}(z) \, \Delta(\sigma,\tau) \; , \tag{3.21a}$$
$$\big\langle a^m(\sigma) a^n(\tau) \big\rangle = \beta\hbar \, g^{mn}(z) \, \delta(\sigma-\tau) \; , \tag{3.21b}$$
$$\big\langle b^m(\sigma) c^n(\tau) \big\rangle = -2\beta\hbar \, g^{mn}(z) \, \delta(\sigma-\tau) \; , \tag{3.21c}$$
$$\big\langle \xi^a_{\rm qu}(\sigma) \overline{\xi}^b_{\rm qu}(\tau) \big\rangle = \delta^{ab} \, \theta(\sigma-\tau) \; , \tag{3.21d}$$
$$\big\langle \xi^a_{\rm qu}(\sigma) \xi^b_{\rm qu}(\tau) \big\rangle = 0 = \big\langle \overline{\xi}^a_{\rm qu}(\sigma) \overline{\xi}^b_{\rm qu}(\tau) \big\rangle \; , \tag{3.21e}$$

where the $\delta(\sigma-\tau)$ is the "Kronecker delta", and $-1 \leq \tau, \sigma \leq 0$. The definitions of various functions are defined as $\Delta(\sigma,\tau) = \sigma(\tau+1)\theta(\sigma-\tau) + \tau(\sigma+1)\theta(\tau-\sigma) = \Delta(\tau,\sigma)$, $\theta(\tau-\tau) = \frac{1}{2}$, $\theta(\tau-\sigma) = -\theta(\sigma-\tau)+1$, and so forth, which we list in (B.1) (see also [26]).

3.4 Explicit form of the transition element in $\mathcal{N}=1$ system

We can also describe the transition element in the $\mathcal{N}=1$ supersymmetric quantum system in terms of the dynamical bosonic and fermionic fields and free Majorana fields (see eq. (2.81) in [26]):

$$\langle\, z, \overline{\eta}, \overline{\eta}_{\rm gh} \,|\, \exp\Big(-\frac{\beta}{\hbar}\widehat{\mathscr{H}}{}^1_H\Big) | \, y, \zeta, \zeta_{\rm gh} \,\rangle = \bigg(\frac{g(z)}{g(y)}\bigg)^{\frac{1}{4}} \frac{1}{(2\pi\beta\hbar)^{D/2}} \, {\rm e}^{\overline{\eta}_a \zeta^a} \, {\rm e}^{\overline{\eta}_{\rm gh}\cdot\zeta_{\rm gh}} \big\langle {\rm e}^{-\frac{1}{\hbar}S^{(\rm int)}_{1,H}} \big\rangle, \tag{3.22a}$$

$$\begin{aligned}
\overline{\eta}_a \zeta^a + \overline{\eta}_{\rm gh}\cdot\zeta_{\rm gh} - \frac{1}{\hbar}S^{(\rm int)}_{1,H} &= -\frac{1}{\hbar}\big(S_{1,H} - S^{(0)}_1\big), \qquad\qquad (3.22\text{b}) \\
-\frac{1}{\hbar}S_{1,H} &= \overline{\eta}_a \zeta^a + \overline{\eta}_{\rm gh}\cdot\zeta_{\rm gh} \\
&\quad - \frac{1}{\beta\hbar}\int_{-1}^0 {\rm d}\tau \, \frac{1}{2} g_{mn}(x) \bigg(\frac{{\rm d}x^m}{{\rm d}\tau}\frac{{\rm d}x^n}{{\rm d}\tau} + b^m c^n + a^m a^n \bigg) \\
&\quad - \int_{-1}^0 {\rm d}\tau \bigg(\delta_{ab} \, \overline{\xi}^a_{\rm qu} \frac{\rm d}{{\rm d}\tau} \xi^b_{\rm qu} + \hat{c}^\dagger_{i,{\rm qu}} \frac{\rm d}{{\rm d}\tau} \hat{c}^i_{\rm qu} \bigg) \\
&\quad - \frac{1}{2}\int_{-1}^0 {\rm d}\tau \, \frac{{\rm d}x^m}{{\rm d}\tau} \omega_{-mab}(x) \, \psi^a_1 \psi^b_1 \\
&\quad - \int_{-1}^0 {\rm d}\tau \, \frac{{\rm d}x^m}{{\rm d}\tau} A^\alpha_m(x) \, \big(\overline{\xi}_{\rm gh} \, T_\alpha \, \xi_{\rm gh}\big) \\
&\quad + \frac{\beta\hbar}{2}\int_{-1}^0 {\rm d}\tau \, F^\alpha_{mn}(x) \, \psi^m_1 \psi^n_1 \, \big(\overline{\xi}_{\rm gh} \, T_\alpha \, \xi_{\rm gh}\big) \\
&\quad - \frac{\beta\hbar}{8}\int_{-1}^0 {\rm d}\tau \, \mathcal{G}_1(x) \; , \qquad\qquad (3.22\text{c}) \\
\mathcal{G}_1(x) &\equiv g^{mn}(x) \Big\{ \Gamma^p_{0mq}(x)\Gamma^q_{0np}(x) + \frac{1}{2}\omega_{-mab}(x)\omega_{-n}{}^{ab}(x) \Big\} \\
&\quad - \frac{1}{3}H_{mnp}(x)H^{mnp}(x) \; , \qquad\qquad (3.22\text{d}) \\
-\frac{1}{\hbar}S^{(0)}_1 &= -\int_{-1}^0 {\rm d}\tau \bigg(\frac{1}{2\beta\hbar} g_{mn}(z) \Big\{ \frac{{\rm d}q^m}{{\rm d}\tau}\frac{{\rm d}q^n}{{\rm d}\tau} + b^m c^n + a^m a^n \Big\} \\
&\quad + \Big\{ \delta_{ab} \, \overline{\xi}^a_{\rm qu} \frac{\rm d}{{\rm d}\tau} \xi^b_{\rm qu} + \hat{c}^\dagger_{i,{\rm qu}} \frac{\rm d}{{\rm d}\tau} \hat{c}^i_{\rm qu} \Big\} \bigg) \; . \qquad\qquad (3.22\text{e})
\end{aligned}$$

In the same way as (3.18), the dynamical fields are decomposed into the background fields and the quantum fields

$$\psi_1^a(\tau) = \psi_{1,\mathrm{bg}}^a(\tau) + \psi_{1,\mathrm{qu}}^a(\tau) \, , \quad \psi_{1,\mathrm{bg}}^a(\tau) = \frac{1}{\sqrt{2}}\big(\zeta^a + \overline{\eta}^a\big) \, , \tag{3.23a}$$

$$\xi_{\mathrm{gh}}^i(\tau) = \zeta_{\mathrm{gh}}^i + \hat{c}_{\mathrm{qu}}^i(\tau) \, , \qquad \overline{\xi}_{i,\mathrm{gh}}(\tau) = \overline{\eta}_{i,\mathrm{gh}} + \hat{c}_{i,\mathrm{qu}}^\dagger(\tau) \, . \tag{3.23b}$$

Notice that the metric in $S_1^{(0)}$ is given at the point z, not at the intermediate point x, while the metric, spin connection, and the fluxes in $S_{1,H}$ are given at the intermediate point x. In the same analogy as the $\mathcal{N}=2$ system, we introduce the bosonic and fermionic propagators. The propagators with respect to the bosonic quantum fields q^m and the ghost fields b^m, c^m and a^m are same as the ones (3.21) in the $\mathcal{N}=2$ system. Here we newly introduce the propagators with respect to the real fermion $\psi_{1,\mathrm{qu}}^a$ given by the combination with two Dirac fermions (3.23a). Since we have already introduced the propagators with respect to the Dirac (complex) fermions ξ_{qu}^a, we can derive the propagators of $\psi_{1,\mathrm{qu}}^a$ in such a way as

$$\big\langle \psi_{1,\mathrm{qu}}^a(\sigma)\psi_{1,\mathrm{qu}}^b(\tau)\big\rangle = \frac{1}{2}\delta^{ab}\Big(\theta(\sigma-\tau) - \theta(\tau-\sigma)\Big) \, . \tag{3.24}$$

The propagator of ghost field $\hat{c}_{\mathrm{gh}}^i$ is also given as

$$\big\langle \hat{c}_{\mathrm{qu}}^i(\sigma)\hat{c}_{j,\mathrm{qu}}^\dagger(\tau)\big\rangle = \delta_j^i\,\theta(\sigma-\tau) \, . \tag{3.25}$$

4. Witten index in $\mathcal{N}=1$ quantum mechanics

In this section we will discuss the Witten index in the $\mathcal{N}=1$ quantum mechanical system derived from the path integral formalism. To obtain this, we will analyze Feynman path integral in terms of Feynman (dis)connected graphs. Since the form of the Witten index (or equivalently, the Dirac index) is same as the one of the chiral anomaly, we refer to the derivation of the chiral anomaly given in section 6.1 and 6.2 of [26].

4.1 Formulation

As mentioned before, by using the identification between the Clifford algebra on the target geometry and the anti-commutation relations of fermions in the quantum mechanics, we can describe the Dirac index equipped with the regulator $\mathscr{R}$ in terms of the transition element of $\mathcal{N}=1$ quantum mechanics

$$\begin{aligned}\mathrm{index}\not{D}(\hat{\omega}) &\equiv \lim_{\beta\to 0}\mathrm{Tr}\big\{\Gamma_{(5)}\mathrm{e}^{-\beta\mathscr{R}}\big\} = \lim_{\beta\to 0}\mathrm{Tr}\big\{(-1)^F\mathrm{e}^{-\frac{\beta}{\hbar}\widehat{\mathscr{H}}_H^1}\big\} \\ &= \lim_{\beta\to 0}\frac{(-i)^{D/2}}{2^{D/2}}\,\mathrm{Tr}\prod_{a=1}^{D}\big(\widehat{\varphi}^a + \widehat{\overline{\varphi}}^a\big)\,\mathrm{e}^{-\frac{\beta}{\hbar}\widehat{\mathscr{H}}_H^1} \, .\end{aligned} \tag{4.1}$$

Note that the chirality operator $\Gamma_{(5)}$ on the target geometry can be identified with the fermion number operator $(-1)^F$ in the $\mathcal{N}=1$ quantum mechanics, i.e., the chirality operator is defined as $\Gamma_{(5)} = (-i)^{D/2}\Gamma^1\Gamma^2\cdots\Gamma^D$, the number operator $(-1)^F$ is replaced in terms of the fermion operators

$$\Gamma^a \equiv \sqrt{2}\psi_1^a = \left(\widehat{\varphi}^a + \widehat{\overline{\varphi}}^a\right)\,, \quad \Gamma_{(5)} \equiv (-i)^{D/2}\prod_{a=1}^{D}\left(\widehat{\varphi}^a + \widehat{\overline{\varphi}}^a\right)\,. \tag{4.2}$$

Notice that the fermion ψ_2^a, which is now included in the path integral measure while does not appear in the Hamiltonian, has dimension $2^{D/2}$. Then we should divide by $2^{D/2}$ from the formulation $(-i)^{D/2}\prod_{a=1}^{D}(\widehat{\varphi}^a + \widehat{\overline{\varphi}}^a)$ by hand. (See the explanation in section 6.1 in [26] and we will find that this factor is canceled out via the fermionic measure computation.) The symbol Tr in the above expression of the index is defined as

$$\mathrm{Tr}\,\mathcal{O} \equiv \int \mathrm{d}^D x_0 \sqrt{g(x_0)}\prod_{a=1}^{D}\left(\mathrm{d}\zeta^a \mathrm{d}\overline{\zeta}_a\right)\mathrm{e}^{\overline{\zeta}\zeta}\,\langle\, x_0, \overline{\zeta}\,|\mathcal{O}|\, x_0, \zeta\,\rangle\,. \tag{4.3}$$

Then, inserting the complete set of the fermion coherent states (3.5), we obtain the explicit form of the Dirac index, i.e., the Witten index with respect to the $\mathcal{N}=1$ quantum mechanical path integral:

$$\begin{aligned}\text{index}\, \not{D}(\hat{\omega}) = \lim_{\beta\to 0}\frac{(-i)^{D/2}}{2^{D/2}}&\int \mathrm{d}^D x_0\sqrt{g(x_0)}\prod_{a=1}^{D}\Big(\mathrm{d}\overline{\eta}_a \mathrm{d}\eta^a \mathrm{d}\zeta^a \mathrm{d}\overline{\zeta}_a\Big)\mathrm{e}^{\overline{\zeta}\zeta}\langle\,\overline{\zeta}\,|\prod_{b=1}^{D}\left(\widehat{\varphi}^b + \widehat{\overline{\varphi}}^b\right)|\,\eta\,\rangle\,\mathrm{e}^{-\overline{\eta}\eta} \\ &\times\langle\, x_0, \overline{\eta}\,|\exp\Big(-\frac{\beta}{\hbar}\widehat{\mathscr{H}}{}^1_H\Big)|\, x_0, \zeta\,\rangle\,. \end{aligned}\tag{4.4a}$$

Here the appearing transition element has already described in the previous section such as

$$\langle\, x_0, \overline{\eta}\,|\exp\Big(-\frac{\beta}{\hbar}\widehat{\mathscr{H}}{}^1_H\Big)|\, x_0, \zeta\,\rangle = \frac{1}{(2\pi\beta\hbar)^{D/2}}\,\mathrm{e}^{\overline{\eta}_a\zeta^a}\Big\langle \exp\Big(-\frac{1}{\hbar}S^{(\mathrm{int})}_{1,H}\Big)\Big\rangle\,, \tag{4.4b}$$

$$\begin{aligned}-\frac{1}{\hbar}S^{(\mathrm{int})}_{1,H} = &-\frac{1}{2\beta\hbar}\int_{-1}^{0}\mathrm{d}\tau\,\Big\{g_{mn}(x) - g_{mn}(x_0)\Big\}\Big(\dot{q}^m\dot{q}^n + b^m c^n + a^m a^n\Big) \\ &-\frac{1}{2}\int_{-1}^{0}\mathrm{d}\tau\,\dot{q}^m\omega_{-mab}(x)\,\psi_1^{ab} - \frac{\beta\hbar}{8}\int_{-1}^{0}\mathrm{d}\tau\,\mathcal{G}_1(x)\,,\end{aligned} \tag{4.4c}$$

where $x = x_0 + q$, $\omega_{-mab}(x) = \omega_{mab}(x) - H_{mab}(x)$ and $\psi_1^a = \psi_{1,\mathrm{bg}}^a + \psi_{1,\mathrm{qu}}^a(\tau)$. The functional $\mathcal{G}_1(x)$ is defined in (3.22d). The fermionic terms are summarized as

$$\begin{aligned}\int\prod_{a=1}^{D}\Big(\mathrm{d}\overline{\eta}_a \mathrm{d}\eta^a \mathrm{d}\zeta^a \mathrm{d}\overline{\zeta}_a\Big)\mathrm{e}^{\overline{\zeta}\zeta - \overline{\eta}\eta}\langle\,\overline{\zeta}\,|\prod_{b=1}^{D}\left(\widehat{\varphi}^b + \widehat{\overline{\varphi}}^b\right)|\,\eta\,\rangle &= \int\prod_{a=1}^{D}\Big(\mathrm{d}\overline{\eta}_a \mathrm{d}\eta^a \mathrm{d}\zeta^a \mathrm{d}\overline{\zeta}_a\Big)\mathrm{e}^{\overline{\zeta}\zeta - \overline{\eta}\eta + \overline{\zeta}\eta}\prod_{b=1}^{D}(\eta^b + \overline{\zeta}^b) \\ &= \int\prod_{a=1}^{D}\Big(\mathrm{d}\overline{\eta}_a \mathrm{d}\eta^a \mathrm{d}\zeta^a \mathrm{d}\overline{\zeta}_a\Big)\mathrm{e}^{\overline{\zeta}\zeta - \overline{\eta}\eta}\prod_{b=1}^{D}(\eta^b + \overline{\zeta}^b).\end{aligned} \tag{4.5a}$$

The last factor becomes a fermionic delta function $\delta(\eta + \overline{\zeta})$, hence $\langle\,\overline{\zeta}\,|\,\eta\,\rangle = \mathrm{e}^{\overline{\zeta}\eta}$ can be replaced by unity. For the same reason, we rewrite other exponential factor in such a way as $\overline{\zeta}\zeta - \overline{\eta}\eta = -\frac{1}{2}(\eta - \overline{\zeta})(\zeta - \overline{\eta})$. Let us see the measure:

$$\prod_{a=1}^{D} \mathrm{d}\overline{\eta}_a \mathrm{d}\eta^a \mathrm{d}\zeta^a \mathrm{d}\overline{\zeta}_a = \prod_a \mathrm{d}\overline{\eta}_a \mathrm{d}\zeta^a \cdot 2^D \mathrm{d}(\overline{\zeta} + \eta)^D \cdots \mathrm{d}(\overline{\zeta} + \eta)^1 \mathrm{d}(\eta - \overline{\zeta})^1 \cdots \mathrm{d}(\eta - \overline{\zeta})^D \,. \tag{4.5b}$$

Thus, combining the above two equations, we show

$$\begin{aligned} &\int \prod_a \mathrm{d}\overline{\eta}_a \mathrm{d}\zeta^a \Big[2^D \mathrm{d}(\overline{\zeta}{+}\eta)^D \cdots \mathrm{d}(\overline{\zeta}{+}\eta)^1 \mathrm{d}(\eta - \overline{\zeta})^1 \cdots \mathrm{d}(\eta - \overline{\zeta})^D \Big] \, \mathrm{e}^{-\frac{1}{2}(\eta - \overline{\zeta})(\zeta - \overline{\eta})} \prod_b \big(\eta^b + \overline{\zeta}^b \big) \\ &\quad = \int \prod_a \mathrm{d}\overline{\eta}_a \mathrm{d}\zeta^a \prod_b \big(\zeta^b - \overline{\eta}^b \big) \,. \end{aligned} \tag{4.5c}$$

This is again the fermionic delta function, which annihilates the exponential factor $\mathrm{e}^{\overline{\eta}\zeta}$ from the Weyl-ordered Hamiltonian. We perform this fermionic delta function to the transition element. Generically we consider the following equation in the $\mathcal{N} = 1$ system:

$$\int \prod_a \mathrm{d}\overline{\eta}_a \mathrm{d}\zeta^a \prod_b \big(\zeta^b - \overline{\eta}^b \big) \, \mathrm{e}^{\overline{\eta}\zeta} F\Big(\frac{\zeta + \overline{\eta}}{\sqrt{2}} \Big) = 2^{D/2} \int \prod_a \mathrm{d}\psi^a_{1,\mathrm{bg}} \, F(\psi^a_{1,\mathrm{bg}}) \,. \tag{4.5d}$$

The factor $2^{D/2}$ cancels the factor $2^{-D/2}$ in (4.4), which we introduced caused by the free fermion ψ^a_2. Next, rescaling the fermions ψ^a_1 by a factor $(\beta\hbar)^{-\frac{1}{2}}$ as $\psi^a_1 \to (\beta\hbar)^{-\frac{1}{2}} \psi^a_1$, we remove the $\beta\hbar$ dependence in the path integral measure. Here we show the Witten index in the path integral formalism:

$$\mathrm{index} \not{D}(\hat{\omega}) = \lim_{\beta \to 0} \frac{(-i)^{D/2}}{(2\pi)^{D/2}} \int \mathrm{d}^D x_0 \sqrt{g(x_0)} \prod_{a=1}^{D} \mathrm{d}\psi^a_{1,\mathrm{bg}} \Big\langle \exp\Big(-\frac{1}{\hbar} S^{(\mathrm{int})}_{1,H} \Big) \Big\rangle \,, \tag{4.6a}$$

$$\begin{aligned} -\frac{1}{\hbar} S^{(\mathrm{int})}_{1,H} &= -\frac{1}{\beta\hbar} \int_{-1}^{0} \mathrm{d}\tau \, \frac{1}{2} \Big\{ g_{mn}(x) - g_{mn}(x_0) \Big\} \Big(\dot{q}^m \dot{q}^n + b^m c^n + a^m a^n \Big) \\ &\quad - \frac{1}{2\beta\hbar} \int_{-1}^{0} \mathrm{d}\tau \, \dot{q}^m \omega_{-mab}(x) \big(\psi_{1,\mathrm{bg}} + \psi_{1,\mathrm{qu}} \big)^{ab} - \frac{\beta\hbar}{8} \int_{-1}^{0} \mathrm{d}\tau \, \mathcal{G}_1(x) \,, \end{aligned} \tag{4.6b}$$

where $x = x_0 + q$. In addition, all the bosonic and fermionic propagators are proportional to $\beta\hbar$:

$$\big\langle q^m(\sigma) q^n(\tau) \big\rangle = -\beta\hbar \, g^{mn}(x_0) \, \Delta(\sigma, \tau) \,, \tag{4.7a}$$

$$\big\langle q^m(\sigma) \dot{q}^n(\tau) \big\rangle = -\beta\hbar \, g^{mn}(x_0) \, \Big(\sigma + \theta(\tau - \sigma) \Big) \,, \tag{4.7b}$$

$$\big\langle \dot{q}^m(\sigma) \dot{q}^n(\tau) \big\rangle = -\beta\hbar \, g^{mn}(x_0) \, \Big(1 - \delta(\tau - \sigma) \Big) \,, \tag{4.7c}$$

$$\big\langle a^m(\sigma) a^n(\tau) \big\rangle = \beta\hbar \, g^{mn}(x_0) \, \delta(\sigma - \tau) \,, \tag{4.7d}$$

$$\big\langle b^m(\sigma) c^n(\tau) \big\rangle = -2\beta\hbar \, g^{mn}(x_0) \, \delta(\sigma - \tau) \,, \tag{4.7e}$$

$$\big\langle \psi^a_{1,\mathrm{qu}}(\sigma) \psi^b_{1,\mathrm{qu}}(\tau) \big\rangle = \frac{1}{2} \beta\hbar \, \delta^{ab} \Big(\theta(\sigma - \tau) - \theta(\tau - \sigma) \Big) \,. \tag{4.7f}$$

The properties of these functions are seen in (B.1). In the end of the evaluation of the path integral, we should take a limit $\beta \to 0$. There are a number of comments to verify the path integral:

- Disconnected graphs should contribute to the functional integrals, called the Feynman amplitudes [22, 26].
- Graphs of higher order in $\beta\hbar$ do not contribute to Feynman amplitudes in the vanishing limit $\beta \to 0$.
- Terms linear in the quantum fields $\dot{q}^m$ do not contribute because of the periodic boundary condition $q^m(-1) = q^m(0) = 0$.
- Terms linear in the quantum fields q^m do not contribute because of the periodic boundary condition and the mean-value theorem (3.19a), while the terms linear in $\psi^a_{1,\mathrm{qu}}$ contribute because there are no restrictions on the quantum fermion fields except for $\xi^a_{\mathrm{qu}}(-1) = \overline{\xi}^a_{\mathrm{qu}}(0) = 0$.
- We could, for convenience, choose a frame with $\partial_m g_{pq}(x_0) = 0$, called the Riemann normal coordinate frame. Due to this we find $\partial_m e_n{}^a = \partial_m E_a{}^n = 0$, $\Gamma^p_{0nq}(x_0) = 0$ and $\omega_{mab}(x_0) = 0$. Notice, however that $\partial_p \partial_q e_m{}^a(x_0) \neq 0$, $\partial_m \omega_{nab}(x_0) \neq 0$ and so forth.
- The torsion given by the NS-NS flux H_{mnp} (or, in mathematically equivalent form, the Bismut torsion $T^{(\mathrm{B})}$) is also expanded in the Riemann normal coordinate frame around x_0.
- The Feynman amplitudes should be independent of the target space metric, at least invariant under the rescale of the metric.

The torsion is given by the NS-NS three-form flux H_{mnp}, which is represented in terms of the Bismut torsion $T^{(\mathrm{B})}$ in the supergravity [24]:

$$H_{mnp}(x) = \frac{3}{2} J_m{}^q J_n{}^r J_p{}^s \nabla_{[q} J_{rs]}(x) = \frac{3}{2} J_m{}^q J_n{}^r J_p{}^s \partial_{[q} J_{rs]}(x) \; . \tag{4.8}$$

As mentioned in the above comment, we will take the Riemann normal coordinate frame at the point x_0. At this point we can set the flat metric at the lowest order approximation in the following way:

$$g_{mn}(x_0) = \delta_{mn} \; , \quad \partial_p g_{mn}(x_0) = 0 \; , \quad \partial_p \partial_q g_{mn}(x_0) \neq 0 \; . \tag{4.9}$$

Due to (4.8), and since the complex structure is proportional to the metric, the flux (or the torsion) should be also expanded around the point x_0 with the values

$$H_{mnp}(x_0) = \frac{3}{2} J_m{}^q J_n{}^r J_p{}^s \partial_{[q} J_{rs]}(x_0) = 0 \; , \quad \partial_q H_{mnp}(x_0) \neq 0 \; . \tag{4.10}$$

By using this, the evaluation of the path integral becomes much simpler.

Note that we rewrite the derivative of the spin connection in such a way as

$$\begin{aligned} \partial_n \omega_{-mab}(x_0) \int_{-1}^0 \mathrm{d}\tau \, \dot{q}^m q^n &= -\frac{1}{2} \Big(\partial_m \omega_{-nab}(x_0) - \partial_n \omega_{-mab}(x_0) \Big) \int_{-1}^0 \mathrm{d}\tau \, \dot{q}^m q^n \\ &= \frac{1}{2} R_{abmn}(\omega_-(x_0)) \int_{-1}^0 \mathrm{d}\tau \, q^m \dot{q}^n \\ &= \frac{1}{2} R_{mnab}(\omega_+(x_0)) \int_{-1}^0 \mathrm{d}\tau \, q^m \dot{q}^n \; , \end{aligned} \tag{4.11}$$

where we used the symmetricity on a Riemann tensor with torsion $R_{pqmn}(\omega_-) = R_{mnpq}(\omega_+) - (\mathrm{d}H)_{pqmn}$ and the periodicity of the bosonic quantum fields $q^m(0) = q^m(-1)$. Furthermore we also generalized the derivative to the covariant derivative because now we analyze on a point x_0 on which the torsion free connections vanish: $\Gamma^p_{0mn}(x_0) = \omega_{mab}(x_0) = H_{mab}(x_0) = 0$.

Let us evaluate the functional integral in terms of the bosonic propagators (4.7) at the point x_0. The exponent $\langle \exp(-\frac{1}{\hbar} S^{(\mathrm{int})}_{1,H}) \rangle$ contains both connected and disconnected Feynman graphs. First we analyze connected graphs, then we summarize them to obtain the products of connected graphs. Let us introduce the effective action W_H by $\mathrm{e}^{-\frac{1}{\hbar} W_H} = \langle \exp(-\frac{1}{\hbar} S^{(\mathrm{int})}_{1,H}) \rangle$, which is expanded as

$$-\frac{1}{\hbar} W_H = \sum_{k=1}^{\infty} \frac{1}{k!} \left\langle\!\!\left\langle \Big(- \frac{1}{\hbar} S^{(\mathrm{int})}_{1,H} \Big)^k \right\rangle\!\!\right\rangle , \tag{4.12}$$

where $\langle\!\langle \cdots \rangle\!\rangle$ indicates the value given only by the connected Feynman graphs.

For later discussions, it is also worth mentioning that the volume form and the Riemann curvature two-form are given in terms of the vielbein one-form $e^a = e_m{}^a \mathrm{d}x^m$ in the following way:

$$\mathrm{d}^{2n} x_0 \sqrt{g(x_0)}\, \mathcal{E}^{b_1 \cdots b_{2n}} = e^{b_1} \wedge \cdots \wedge e^{b_{2n}} . \tag{4.13}$$

Furthermore, we also find the following formula:

$$\int \prod_{a=1}^{D} \mathrm{d}\psi^a_{1,\mathrm{bg}}\, \psi^{a_1 \cdots a_D}_{1,\mathrm{bg}} = (-)^{D/2}\, \mathcal{E}^{a_1 a_2 \cdots a_D} . \tag{4.14}$$

The trace of the odd number of the curvature two-form vanishes because the permutation of the two-form is symmetric but the flip of the indices is anti-symmetric $\mathrm{tr}(R^{2k-1}) = 0$.

4.2 Pontrjagin classes

4.2.1 Riemannian manifold

In this case $S^{(\mathrm{int})}_1$ becomes much simpler than (4.6a) because there are no terms from H-flux. The spin connection ω_- is also reduced to ω. We also easily find that the terms equipped with higher derivatives carrying more than three bosonic quantum fields q^m always generate higher-loops Feynman graphs because of the absence of the tadpole graphs. Furthermore, the terms of order in $\beta\hbar$ do not contribute to the final result. Then we truncate $S^{(\mathrm{int})}_1$ in the following way:

$$-\frac{1}{\hbar} S^{(\mathrm{int})}_1 = -\frac{1}{2\beta\hbar} R_{mn}(\omega(x_0)) \int_{-1}^{0} \mathrm{d}\tau\, q^m \dot{q}^n , \quad R_{mn} \equiv \frac{1}{2} R_{mnab}(\omega(x_0))\, \psi^a_{1,\mathrm{bg}} \psi^b_{1,\mathrm{bg}} , \tag{4.15}$$

where we used (4.11) with $H = \mathrm{d}H = 0$. Then, the path integral form of the Witten index without H-flux is reduced to

$$\mathrm{index} \not{D}(\omega) = \lim_{\beta \to 0} \frac{(-i)^{D/2}}{(2\pi)^{D/2}} \int \mathrm{d}^D x_0 \sqrt{g(x_0)} \prod_{a=1}^{D} \mathrm{d}\psi^a_{1,\mathrm{bg}} \Big\langle \exp \Big(- \frac{1}{2\beta\hbar} R_{mn} \int_{-1}^{0} \mathrm{d}\tau\, q^m \dot{q}^n \Big) \Big\rangle . \tag{4.16}$$

Let us first evaluate the sum of connected graphs:

$$-\frac{1}{\hbar}W = \log\Big\langle \exp\Big(-\frac{1}{2\beta\hbar}R_{mn}\int_{-1}^{0}\mathrm{d}\tau\, q^m \dot{q}^n\Big)\Big\rangle$$
$$= \sum_{k=1}^{\infty}\frac{1}{k!}\Big(-\frac{1}{2\beta\hbar}\Big)^k R_{m_1 n_1}\cdots R_{m_k n_k}\int_{-1}^{0}\mathrm{d}\tau_1\cdots\mathrm{d}\tau_k \Big\langle\!\Big\langle \big(q^{m_1}\dot{q}^{n_1}\big)(\tau_1)\cdots\big(q^{m_k}\dot{q}^{n_k}\big)(\tau_k)\Big\rangle\!\Big\rangle. \tag{4.17}$$

Since the two indices in the Riemann tensors are anti-symmetric whereas the propagators are symmetric with respect to the exchanging of bosonic quantum fields, we easily find that the contraction at the same "time" τ_i yields a vanishing amplitude. We also know that the partial integration is allowed since $q^m(\tau_i) = 0$ at the end points. Then, there are $(k-1)!$ ways to contract k vertices and the symmetry of each vertex in both q yields a factor 2^{k-1}. Then we find that the effective action (4.17) is described as

$$-\frac{1}{\hbar}W = \sum_{k=1}^{\infty}\frac{1}{k!}\Big(-\frac{1}{2\beta\hbar}\Big)^k (k-1)!2^{k-1}\big(-\beta\hbar\big)^k \cdot R_{m_1 n_1}R_{m_2 n_2}\cdots R_{m_k n_k}g^{n_1 m_2}g^{n_2 m_3}\cdots g^{n_k m_1}$$
$$\times\int_{-1}^{0}\mathrm{d}\tau_1\cdots\mathrm{d}\tau_k \partial_{\tau_1}\Delta(\tau_1,\tau_2)\partial_{\tau_2}\Delta(\tau_2,\tau_3)\cdots\partial_{\tau_{k-1}}\Delta(\tau_{k-1},\tau_k)\partial_{\tau_k}\Delta(\tau_k,\tau_1)$$
$$\equiv \frac{1}{2}\sum_{k=2}^{\infty}\frac{1}{k}\mathrm{tr}(R^k) I_k\ , \tag{4.18a}$$
$$I_k \equiv \int_{-1}^{0}\mathrm{d}\tau_1\cdots\mathrm{d}\tau_k\big[\tau_2 + \theta(\tau_1-\tau_2)\big]\big[\tau_3+\theta(\tau_2-\tau_3)\big]\cdots\big[\tau_1+\theta(\tau_k-\tau_1)\big]\ , \tag{4.18b}$$

where we used $\mathrm{tr}R^1 = 0$. By using the formula (see appendix A.4 in [25])

$$\sum_{k=2}^{\infty}\frac{y^k}{k}I_k = \log\frac{y/2}{\sinh(y/2)} = -\frac{1}{3!}\Big(\frac{y}{2}\Big)^2 + \cdots\ , \tag{4.19}$$

we summarize the form of the effective action

$$-\frac{1}{\hbar}W = \frac{1}{2}\mathrm{tr}\log\Big(\frac{R/2}{\sinh(R/2)}\Big)\ . \tag{4.20}$$

Furthermore, in order to remove the overall factor in front of the path integral (4.4), we rescale the background fermions $\psi^a_{1,\mathrm{bg}} \to \sqrt{\frac{-i}{2\pi}}\psi^a_{1,\mathrm{bg}}$. Then we obtain the path integral form of the Witten index in such a way as

$$\text{index}\ \not{D}(\omega) = \int \mathrm{d}^D x_0\sqrt{g(x_0)}\prod_{a=1}^{D}\mathrm{d}\psi^a_{1,\mathrm{bg}}\ \exp\left[\frac{1}{2}\mathrm{tr}\log\left(\frac{-iR/4\pi}{\sinh(-iR/4\pi)}\right)\right]\ , \tag{4.21a}$$
$$\mathrm{tr}(R^k) = R_{m_1 n_1}R_{m_2 n_2}\cdots R_{m_k n_k}\, g^{n_1 m_2}g^{n_2 m_3}\cdots g^{n_k m_1}\ . \tag{4.21b}$$

Due to the property of $\mathrm{tr}(R^k)$, this value becomes zero when $D = 4k+2$. Let us simplify the formula (4.21) by integrating the background fermion $\psi^a_{1,\mathrm{bg}}$ of (4.21) with noticing the formulae (4.13) (in particular, eq. (4.14)):

$$\text{index}\ \not{D}(\omega) = \int_{\mathcal{M}}\exp\left[\frac{1}{2}\mathrm{tr}\log\left(\frac{iR/4\pi}{\sinh(iR/4\pi)}\right)\right]\ , \qquad R_{mn} = \frac{1}{2}R_{mnab}(\omega)\, e^a\wedge e^b\ . \tag{4.22}$$

This is the well-known form of Dirac index on the Riemannian manifold $\mathcal{M}$. The integrand is called the (Dirac) $\hat{\mathcal{A}}$-genus.

4.2.2 Torsional manifold

This case is still simple. Since there does not exist an interaction term with single quantum fermion, all the Feynman amplitudes are of order in $(\beta\hbar)^k$, where k is a non-negative integer. Thus, since we are interested only in the amplitudes of order in $(\beta\hbar)^0$ which remain in the vanishing limit $\beta \to 0$, we can neglect the last term in (4.6b) which yields graphs of higher order in $\beta\hbar$. We can also neglect the interaction terms including more than three quantum fields which yield more than two-loops graphs. Thus we truncate $S_{1,H}^{(\mathrm{int})}$ carrying only two bosonic and fermionic quantum fields to

$$-\frac{1}{\hbar}S_{1,H}^{(\mathrm{int})} = -\frac{1}{2\beta\hbar}R_{mn}^{(+)}\int_{-1}^{0}\mathrm{d}\tau\, q^m \dot{q}^n \ , \tag{4.23a}$$

where we used (4.11) with $\mathrm{d}H = 0$ and

$$R_{mn}^{(+)} \equiv \frac{1}{2}R_{mnab}(\omega_+(x_0))\psi_{1,\mathrm{bg}}^a \psi_{1,\mathrm{bg}}^b \ . \tag{4.23b}$$

The effective action, or the functional integral of the connected graphs are given in terms of (4.12):

$$\begin{aligned}-\frac{1}{\hbar}W_H &= \sum_{N=1}^{\infty}\frac{1}{N!}\Big\langle\!\!\Big\langle\Big(-\frac{1}{\hbar}S_1^{(\mathrm{int})}\Big)^N\Big\rangle\!\!\Big\rangle = \sum_{N=1}^{\infty}\frac{1}{N!}\Big(-\frac{1}{2\beta\hbar}\Big)^N\Big\langle\!\!\Big\langle\Big(R_{mn}^{(+)}\int_{-1}^{0}\mathrm{d}\tau q^m\dot{q}^n\Big)^N\Big\rangle\!\!\Big\rangle \qquad (4.24)\\ &= \sum_{N=1}^{\infty}\frac{1}{N!}\Big(-\frac{1}{2\beta\hbar}\Big)^N R_{m_1n_1}^{(+)}\cdots R_{m_Nn_N}^{(+)}\int_{-1}^{0}\mathrm{d}\tau_1\cdots\mathrm{d}\tau_N\Big\langle\!\!\Big\langle(q^{m_1}\dot{q}^{n_1})(\tau_1)\cdots(q^{m_N}\dot{q}^{n_N})(\tau_N)\Big\rangle\!\!\Big\rangle,\end{aligned}$$

where we abbreviated $\psi_{1,\mathrm{qu}}^a \equiv \psi^a$. This is exactly same equation as (4.17) except for the Riemann curvature tensors. Then, after the rescaling of the background fermion fields, the result is given by (4.21) in the following way:

$$\begin{aligned}\mathrm{index}\, \not{D}(\hat{\omega}) &= \int \mathrm{d}^D x_0 \sqrt{g(x_0)}\prod_{a=1}^{D}\mathrm{d}\psi_{1,\mathrm{bg}}^a\, \exp\left[\frac{1}{2}\mathrm{tr}\log\left(\frac{-iR^{(+)}/4\pi}{\sinh(-iR^{(+)}/4\pi)}\right)\right] , &(4.25a)\\ \mathrm{tr}(R_{(+)}^k) &= R_{m_1n_1}^{(+)}R_{m_2n_2}^{(+)}\cdots R_{m_kn_k}^{(+)}\, g^{n_1m_2}g^{n_2m_3}\cdots g^{n_km_1} \ , &(4.25b)\\ R_{mn}^{(+)} &\equiv \frac{1}{2}R_{mnab}(\omega_+(x_0))\psi_{1,\mathrm{bg}}^a\psi_{1,\mathrm{bg}}^b \ . &(4.25c)\end{aligned}$$

The most significant point is that we obtained the same result which appears in the Mavromatos' work [20−22]. This is exactly same equation as (4.17) except for the Riemann curvature tensors. Then, after the rescaling of the background fermion fields, the result is given in the following way: Finally, let us integrate the background fermion $\psi_{1,\mathrm{bg}}^a$ of (4.25) in the same analogy as (4.22):

$$\begin{aligned}\mathrm{index}\, \not{D}(\hat{\omega}) &= \int \exp\left[\frac{1}{2}\mathrm{tr}\log\left(\frac{iR^{(+)}/4\pi}{\sinh(iR^{(+)}/4\pi)}\right)\right], &(4.26a)\\ \mathrm{tr}(R_{(+)}^k) &= R_{m_1n_1}^{(+)}R_{m_2n_2}^{(+)}\cdots R_{m_kn_k}^{(+)}\, g^{n_1m_2}g^{n_2m_3}\cdots g^{n_km_1}, &(4.26b)\\ R_{mn}^{(+)} &\equiv \frac{1}{2}R_{mnab}(\omega_+)e^a\wedge e^b. &(4.26c)\end{aligned}$$

5. $\mathcal{N} = 1$ quantum mechanics for internal gauge symmetry

In this section we will focus on the gauge field and the invariant polynomial derived from the path integral. The transition element is described in terms of the quantum Hamiltonian in (2.6). Since the $\hat{c}$-ghost field in (2.6) are independent of the other fields, the path integral of this $\hat{c}$-ghost can be evaluated on a flat geometry and can be applied to an arbitrary curved manifold. Thus let us first formulate the path integral of this ghost field on a flat geometry, and we apply this result on the computation on a generic curved geometry. Here we again follow the convention in [26].

5.1 Formulation

The Dirac index is given by the Witten index in a following way:

$$\text{index}\ \not{D}(\hat{\omega}, A) \equiv \lim_{\beta\to 0} \mathrm{Tr}'\big\{(-1)^F \mathrm{e}^{-\frac{\beta}{\hbar}\widehat{\mathscr{H}}{}^1_H}\big\} = \lim_{\beta\to 0} \frac{(-i)^{D/2}}{2^{D/2}} \mathrm{Tr} \prod_{a=1}^{D} \big(\widehat{\varphi}^a + \widehat{\overline{\varphi}}^a\big) P_{\mathrm{gh}} \mathrm{e}^{-\frac{\beta}{\hbar}\widehat{\mathscr{H}}{}^1_H}, \tag{5.1a}$$

$$P_{\mathrm{gh}} \equiv \, : x\, \mathrm{e}^{-x} : \, , \qquad x \equiv \hat{c}_i^{\dagger} \hat{c}^i \, , \tag{5.1b}$$

where we expressed the trace with prime in order to evaluate the trace only over the one-particle ghost sector. We also introduce the one-particle ghost "projection operator" P_{gh} instead of the trace with prime. We should also define the completeness relation of the fermionic states as

$$I_{\mathrm{gh}} \equiv \int \prod_{i=1}^{\dim R} \mathrm{d}\overline{\eta}_{i,\mathrm{gh}} \mathrm{d}\eta^i_{\mathrm{gh}} \, |\, \eta_{\mathrm{gh}} \,\rangle \, \mathrm{e}^{-\overline{\eta}_{\mathrm{gh}} \cdot \eta_{\mathrm{gh}}} \langle \, \overline{\eta}_{\mathrm{gh}} \,| \, , \quad I_{\mathrm{f}} \equiv \int \prod_{a=1}^{D} \mathrm{d}\overline{\eta}_{a,\mathrm{f}} \mathrm{d}\eta^a_{\mathrm{f}} \, |\, \eta_{\mathrm{f}} \,\rangle \, \mathrm{e}^{-\overline{\eta}_{\mathrm{f}} \cdot \eta_{\mathrm{f}}} \langle \, \overline{\eta}_{\mathrm{f}} \,| \, . \tag{5.2}$$

The trace formulae for the ghost and physical fermionic states are also independently defined by

$$\mathrm{tr}_{\mathrm{gh}} \mathcal{O} \equiv \int \prod_{i=1}^{\dim R} \mathrm{d}\chi^i_{\mathrm{gh}} \mathrm{d}\overline{\chi}_{i,\mathrm{gh}} \mathrm{e}^{\overline{\chi}_{\mathrm{gh}} \cdot \chi_{\mathrm{gh}}} \langle \, \overline{\chi}_{\mathrm{gh}} \,| \mathcal{O} |\, \chi_{\mathrm{gh}} \,\rangle \, , \quad \mathrm{tr}_{\mathrm{f}} \mathcal{O} \equiv \int \prod_{a=1}^{D} \mathrm{d}\chi^a_{\mathrm{f}} \mathrm{d}\overline{\chi}_{a,\mathrm{f}} \mathrm{e}^{\overline{\chi}_{\mathrm{f}} \cdot \chi_{\mathrm{f}}} \langle \, \overline{\chi}_{\mathrm{f}} \,| \mathcal{O} |\, \chi_{\mathrm{f}} \,\rangle . \tag{5.3}$$

In a usual case this trace formula gives the anti-periodic boundary condition on the fermion. The fermion number operator $(-1)^F$, which acts on the physical fermion states, flips the condition to the periodic boundary condition (see section 2.4 in [26]). By using these formulae, we rewrite the Dirac index given by (5.1):

$$\begin{aligned} \text{index}\ \not{D}(\hat{\omega}, A) = \lim_{\beta\to 0} \frac{(-i)^{D/2}}{2^{D/2}} & \int \mathrm{d}^D x_0 \sqrt{g(x_0)} \\ & \times \mathrm{tr}_{\mathrm{f}}\, \mathrm{tr}_{\mathrm{gh}} \big\langle \, x_0, \overline{\chi}_{\mathrm{gh}}, \overline{\chi}_{\mathrm{f}} \,\big| \prod_{a=1}^{D} \big(\widehat{\varphi}^a + \widehat{\overline{\varphi}}^a\big) P_{\mathrm{gh}} I_{\mathrm{gh}} I_{\mathrm{f}} \, \mathrm{e}^{-\frac{\beta}{\hbar}\widehat{\mathscr{H}}{}^1_H} \big|\, x_0, \chi_{\mathrm{f}}, \chi_{\mathrm{gh}} \,\big\rangle \, . \end{aligned} \tag{5.4}$$

Of course the ghost Hilbert space and the physical fermion Hilbert space are independent of each other. Then these completeness relation act on the individual spaces without any interruption. Now let us evaluate the trace in the ghost sector:

$$
\begin{aligned}
&\mathrm{tr}_{\mathrm{gh}} \langle\, \overline{\chi}_{\mathrm{gh}} \,| P_{\mathrm{gh}} I_{\mathrm{gh}} \mathrm{e}^{-\frac{\beta}{\hbar}\widehat{\mathscr{H}}^1_H} |\, \chi_{\mathrm{gh}} \,\rangle \\
&\qquad = \int \prod_i \mathrm{d}\chi^i_{\mathrm{gh}} \mathrm{d}\overline{\chi}_{i,\mathrm{gh}}\, \mathrm{e}^{\overline{\chi}_{\mathrm{gh}}\cdot\chi_{\mathrm{gh}}} \prod_j \mathrm{d}\overline{\eta}_{j,\mathrm{gh}} \mathrm{d}\eta^j_{\mathrm{gh}}\, \mathrm{e}^{-\overline{\eta}_{\mathrm{gh}}\cdot\eta_{\mathrm{gh}}} \langle\, \overline{\chi}_{\mathrm{gh}} \,| P_{\mathrm{gh}} |\, \eta_{\mathrm{gh}} \,\rangle \langle\, \overline{\eta}_{\mathrm{gh}} \,| \mathrm{e}^{-\frac{\beta}{\hbar}\widehat{\mathscr{H}}^1_H} |\, \chi_{\mathrm{gh}} \,\rangle \,.
\end{aligned} \tag{5.5}
$$

Since $P_{\mathrm{gh}} =: x\mathrm{e}^{-x}:$ projects the ghost coherent state $|\, \eta_{\mathrm{gh}} \,\rangle$ onto its one-particle part $P_{\mathrm{gh}} |\, \eta_{\mathrm{gh}} \,\rangle = c^\dagger_i \eta^i_{\mathrm{gh}} |\, 0 \,\rangle$, the matrix element of the ghost projection operator P_{gh} is easily computed and yields

$$
\langle\, \overline{\chi}_{\mathrm{gh}} \,| P_{\mathrm{gh}} |\, \eta_{\mathrm{gh}} \,\rangle = \sum_{i=1}^{\dim R} \overline{\chi}_{i,\mathrm{gh}}\, \eta^i_{\mathrm{gh}} = \overline{\chi}_{\mathrm{gh}} \cdot \eta_{\mathrm{gh}} \,. \tag{5.6}
$$

Then we can integrate out the ghost variables η^i_{gh} and $\overline{\chi}_{i,\mathrm{gh}}$ and define a new kind of projection operator in the following way:

$$
\int \prod_i \mathrm{d}\eta^i_{\mathrm{gh}} \mathrm{d}\overline{\chi}_{i,\mathrm{gh}}\, \mathrm{e}^{\overline{\chi}_{\mathrm{gh}}\cdot\chi_{\mathrm{gh}} - \overline{\eta}_{\mathrm{gh}}\cdot\eta_{\mathrm{gh}}} \langle\, \overline{\chi}_{\mathrm{gh}} \,| P_{\mathrm{gh}} |\, \eta_{\mathrm{gh}} \,\rangle = \sum_{i=1}^{\dim R} \prod_{\ell \neq i} \Big(\overline{\eta}_{\ell,\mathrm{gh}} \chi^\ell_{\mathrm{gh}} \Big) \equiv P^{\mathrm{gh}}_{\overline{\eta},\chi} \,. \tag{5.7}
$$

This operator annihilates all terms containing more than two ghost fields $\overline{\eta}_{\mathrm{gh}}$ and χ_{gh}. Because of this we interpret this operator as a kind of "projection operator" onto terms which are linear in $\overline{\eta}_{\mathrm{gh}}$ and χ_{gh}, and onto terms independent of any ghost fields.

By using (4.5), (5.7) and (3.22), and rescaling physical fermions as $\psi^a_1 \to (\beta\hbar)^{-\frac{1}{2}} \psi^a_1$, while keeping the scale of the ghost fields unchanged, we can evaluate the Dirac index (5.1):

$$
\mathrm{index}\, \not{D}(\hat{\omega}, A) = \lim_{\beta \to 0} \frac{(-i)^{D/2}}{(2\pi)^{D/2}} \int \mathrm{d}^D x_0 \sqrt{g(x_0)} \prod_{i=1}^{\dim R} \mathrm{d}\chi^i_{\mathrm{gh}} \mathrm{d}\overline{\eta}_{i,\mathrm{gh}} P^{\mathrm{gh}}_{\overline{\eta},\chi} \mathrm{e}^{\overline{\eta}_{\mathrm{gh}}\cdot\chi_{\mathrm{gh}}} \prod_{a=1}^{D} \mathrm{d}\psi^a_{1,\mathrm{bg}} \langle \mathrm{e}^{-\frac{1}{\hbar} S^{(\mathrm{int})}_{1,H}} \rangle, \tag{5.8a}
$$

$$
\begin{aligned}
-\frac{1}{\hbar} S^{(\mathrm{int})}_{1,H} &= -\frac{1}{2\beta\hbar} \int_{-1}^{0} \mathrm{d}\tau \Big\{ g_{mn}(x) - g_{mn}(x_0) \Big\} \Big(\dot{q}^m \dot{q}^n + b^m c^n + a^m a^n \Big) \\
&\quad - \frac{1}{2\beta\hbar} \int_{-1}^{0} \mathrm{d}\tau\, \dot{q}^m \omega_{-mab}(x)\, \psi^{ab}_1 - \frac{\beta\hbar}{8} \int_{-1}^{0} \mathrm{d}\tau\, \mathcal{G}_1(x) \\
&\quad - \int_{-1}^{0} \mathrm{d}\tau\, \dot{q}^m A^\alpha_m(x) \big(\overline{\xi}_{\mathrm{gh}} T_\alpha \xi_{\mathrm{gh}} \big) + \frac{1}{2} \int_{-1}^{0} \mathrm{d}\tau\, F^\alpha_{ab}(x)\, \psi^a_1 \psi^b_1 \big(\overline{\xi}_{\mathrm{gh}} T_\alpha \xi_{\mathrm{gh}} \big) \,,
\end{aligned} \tag{5.8b}
$$

with $x = x_0 + q$ and the boundary conditions $q^m(-1) = q^m(0) = 0$, $\int^0_{-1} \mathrm{d}\tau\, q^m(\tau) = 0$, and

$$
\psi^a_{1,\mathrm{qu}}(-1) = \frac{1}{\sqrt{2}} \overline{\xi}^a_{\mathrm{qu}}(-1) \,, \quad \psi^a_{1,\mathrm{qu}}(0) = \frac{1}{\sqrt{2}} \xi^a_{\mathrm{qu}}(0) \,, \quad \hat{c}^i_{\mathrm{qu}}(-1) = \hat{c}^\dagger_{i,\mathrm{qu}}(0) = 0 \,. \tag{5.9}
$$

In addition we can rewrite the expansion of gauge field in such a way as

$$\begin{aligned}
\partial_n A^\alpha_m(x_0) \int_{-1}^0 \mathrm{d}\tau\, \dot{q}^m q^n &= -\frac{1}{2}\Big(\partial_m A^\alpha_n(x_0) - \partial_n A^\alpha_m(x_0)\Big) \int_{-1}^0 \mathrm{d}\tau\, \dot{q}^m q^n \\
&= \frac{1}{2}\Big[F^\alpha_{mn}(x_0) - f^\alpha{}_{\beta\gamma}\, A^\beta_m(x_0) A^\gamma_n(x_0)\Big] \int_{-1}^0 \mathrm{d}\tau\, q^m \dot{q}^n \\
&\equiv \frac{1}{2}\mathcal{F}^\alpha_{mn}(x_0) \int_{-1}^0 \mathrm{d}\tau\, q^m \dot{q}^n \,, \qquad (5.10a)
\end{aligned}$$

$$F^\alpha_{mn}(x_0) = \partial_m A^\alpha_n(x_0) - \partial_n A^\alpha_m(x_0) + f^\alpha{}_{\beta\gamma}\, A^\beta_m(x_0) A^\gamma_n(x_0) \,, \qquad (5.10b)$$

where F^α_{mn} is the field strength of the gauge field and $f^\alpha{}_{\beta\gamma}$ is the structure constant of the gauge group. Notice that the ghost fermions ξ_{gh} and $\overline{\xi}_{\mathrm{gh}}$ obey the anti-periodic boundary condition, while the physical fermions ξ_{f} and $\overline{\xi}_{\mathrm{f}}$ follow the periodic boundary condition because of the insertion of $(-1)^F$. This indicates that any closed-loop graphs of the ghost fields yield zero amplitudes and that only tree graphs contribute to non-vanishing amplitudes. Because of this, disconnected graphs with respect to the $\hat{c}$-ghost amplitudes does not appear in this path integral transition element. This statement is quite strong.

5.2 Chern character

5.2.1 Chern character on flat geometry without H-flux

Let us first consider the simplest system on a flat geometry with vanishing flux $H = \mathrm{d}H = 0$. In this case there are no (background) interaction terms which carries negative powers of $\beta\hbar$, contractions of any physical fields q^m and $\psi^a_{1,\mathrm{qu}}$ become irrelevant under the vanishing limit $\beta \to 0$. Then we can neglect the term linear in $A^\alpha_m(x_0 + q)$ and the path integral (5.8) is reduced to

$$\mathrm{index}\, \not{D}(A) = \lim_{\beta\to 0} \frac{(-i)^{D/2}}{(2\pi)^{D/2}} \int \mathrm{d}^D x_0 \prod_{i=1}^{\dim R} \mathrm{d}\chi^i_{\mathrm{gh}} \mathrm{d}\overline{\eta}_{i,\mathrm{gh}}\, P^{\mathrm{gh}}_{\overline{\eta},\chi}\, \mathrm{e}^{\overline{\eta}_{\mathrm{gh}}\cdot\chi_{\mathrm{gh}}} \prod_{a=1}^{D} \mathrm{d}\psi^a_{1,\mathrm{bg}} \big\langle \mathrm{e}^{-\frac{1}{\hbar}S^{(\mathrm{int})}_{1,H}} \big\rangle \,, \qquad (5.11a)$$

$$-\frac{1}{\hbar}S^{(\mathrm{int})}_{1,H} = (F(x_0))^i{}_j \int_{-1}^0 \mathrm{d}\tau\, \big(\overline{\xi}_{\mathrm{gh}} + \hat{c}^\dagger_{\mathrm{qu}}(\tau)\big)_i \big(\xi_{\mathrm{gh}} + \hat{c}_{\mathrm{qu}}(\tau)\big)^j \,, \qquad (5.11b)$$

where $(F(x_0))^i{}_j = \frac{1}{2} F^\alpha_{ab}(x_0) \psi^{ab}_{1,\mathrm{bg}} (T_\alpha)^i{}_j$. As we mentioned before, we only analyze the ghost tree graphs via the expansion of the above form:

$$\begin{aligned}
&\Big\langle \exp\Big(F^i{}_j \int_{-1}^0 \mathrm{d}\tau\, \big(\overline{\xi}_{\mathrm{gh}} + \hat{c}^\dagger_{\mathrm{qu}}(\tau)\big)_i \big(\xi_{\mathrm{gh}} + \hat{c}_{\mathrm{qu}}(\tau)\big)^j \Big) \Big\rangle \\
&\quad = 1 + \sum_{k=1}^\infty \frac{1}{k!} \overline{\eta}_{j,\mathrm{gh}} \big(F^k\big)^j{}_l\, \chi^l_{\mathrm{gh}} \Big[k! \int_{-1}^0 \mathrm{d}\sigma_1 \cdots \mathrm{d}\sigma_k\, \theta(\sigma_1 - \sigma_2)\, \theta(\sigma_2 - \sigma_3) \cdots \theta(\sigma_{k-1} - \sigma_k) \Big] \\
&\quad = 1 + \sum_{k=1}^\infty \frac{1}{k!} \overline{\eta}_{j,\mathrm{gh}} \big(F^k\big)^j{}_l\, \chi^l_{\mathrm{gh}} \,. \qquad (5.12)
\end{aligned}$$

Note that the factor $k!$ in the square bracket in the second line is due to the fact that we can order the k vertices into a tree in $k!$ ways. We also used the following integral:

$$\int_{-1}^{0} \mathrm{d}\sigma_1 \cdots \mathrm{d}\sigma_k \, \theta(\sigma_1 - \sigma_2) \, \theta(\sigma_2 - \sigma_3) \cdots \theta(\sigma_{k-1} - \sigma_k) = \frac{1}{k!} \, . \tag{5.13}$$

Integral of the ghost fields of (5.12) gives the following simple result:

$$\begin{aligned}
&\int \prod_{i=1}^{\dim R} \mathrm{d}\chi^i_{\mathrm{gh}} \mathrm{d}\overline{\eta}_{i,\mathrm{gh}} \, P^{\mathrm{gh}}_{\overline{\eta},\chi} \, \mathrm{e}^{\overline{\eta}_{\mathrm{gh}} \cdot \chi_{\mathrm{gh}}} \Big\langle \exp\Big(F^i{}_j \int_{-1}^{0} \mathrm{d}\tau \, \big(\overline{\xi}_{\mathrm{gh}} + \hat{c}^\dagger_{\mathrm{qu}}(\tau) \big)_i \, \big(\xi_{\mathrm{gh}} + \hat{c}_{\mathrm{qu}}(\tau) \big)^j \Big) \Big\rangle \\
&\quad = \int \prod_{i=1}^{\dim R} \mathrm{d}\chi^i_{\mathrm{gh}} \mathrm{d}\overline{\eta}_{i,\mathrm{gh}} \, P^{\mathrm{gh}}_{\overline{\eta},\chi} \, \mathrm{e}^{\overline{\eta}_{\mathrm{gh}} \cdot \chi_{\mathrm{gh}}} \bigg(1 + \sum_{k=1}^{\infty} \frac{1}{k!} \overline{\eta}_{j,\mathrm{gh}} \, \big(F^k \big)^j{}_l \, \chi^l_{\mathrm{gh}} \bigg) \\
&\quad = \sum_{j=1}^{\dim R} \bigg[\delta^j{}_j + \sum_{k=1}^{\infty} \frac{1}{k!} \big(F^k \big)^j{}_j \bigg] = \sum_{i=1}^{\dim R} \exp\big(F \big)^i{}_i \equiv \mathrm{Tr}_R \exp\big(F \big) \, ,
\end{aligned} \tag{5.14}$$

where the symbol Tr_R denotes the trace in the R representation of the gauge group. Summarizing the integral and rescaling the background fermion in such a way as $\psi^a_{1,\mathrm{bg}} \to \sqrt{\frac{-i}{2\pi}} \psi^a_{1,\mathrm{bg}}$, we obtain

$$\mathrm{index} \, \not{D}(A) = \int \mathrm{d}^D x_0 \prod_{a=1}^{D} \mathrm{d}\psi^a_{1,\mathrm{bg}} \mathrm{Tr}_R \exp\Big(-\frac{i}{2\pi} F \Big) \, , \quad F = \frac{1}{2} F^\alpha_{ab}(x_0) \, \psi^a_{1,\mathrm{bg}} \psi^b_{1,\mathrm{bg}} \, T_\alpha \, . \tag{5.15}$$

This is nothing but the Chern character of the gauge fields A^α_m. When we explicitly calculate, we should use the formulae (4.13). In the same way as (4.22), let us integrate the background fermions with respect to (4.13) and obtain

$$\mathrm{index} \, \not{D}(A) = \int_{\mathcal{M}} \mathrm{Tr}_R \exp\Big(\frac{i}{2\pi} F \Big) \, , \quad F = \frac{1}{2} F_{ab} \, e^a \wedge e^b = \mathrm{d}A + A \wedge A \, . \tag{5.16}$$

5.2.2 Torsional manifold

Let us easily generalize the equation (5.15) to the one on a curved manifold $\mathcal{M}$ (in the presence of torsion H). Since the Hilbert spaces of the physical states and the $\hat{c}$-ghost states are independent of each other, the functional integrals of the Dirac index are also performed independently. Then, combining the functional integral of the physical field sector (4.21) and the functional integral of the $\hat{c}$-ghost sector (5.15), we obtain the Dirac index in the following representation:

$$\mathrm{index} \, \not{D}(\hat{\omega}, A) = \int_{\mathcal{M}} \exp\bigg[\frac{1}{2} \mathrm{tr} \log\bigg(\frac{iR^{(+)}/4\pi}{\sinh(iR^{(+)}/4\pi)} \bigg) \bigg] \mathrm{Tr}_R \exp\Big(\frac{i}{2\pi} F \Big) \, , \tag{5.17a}$$

$$R^{(+)}_{mn} = R_{mnab}(\omega_+) \, e^a \wedge e^b \, , \quad F = \frac{1}{2} F_{ab} \, e^a \wedge e^b \, . \tag{5.17b}$$

The index on a Riemannian manifold without torsion can be easily obtained when we choose $H = 0$ in this form.

6. Witten index in $\mathcal{N}=2$ quantum mechanics

In this section let us analyze the Euler characteristics on the manifold with torsion H. In the case of vanishing torsion, we will find a form of the Gauss-Bonnet theorem.

6.1 Formulation

The Euler characteristics χ on the target space geometry can also be expressed in terms of the $\mathcal{N}=2$ supersymmetric quantum mechanics (see section 14.3 in [19])

$$\chi \equiv \lim_{\beta\to 0} \mathrm{Tr}\big\{\Gamma_{(5)}\widetilde{\Gamma}_{(5)}\mathrm{e}^{-\beta\mathscr{R}}\big\} = \lim_{\beta\to 0} \mathrm{Tr} \prod_{a=1}^{D} \big(\widehat{\varphi}^a + \widehat{\overline{\varphi}}^a\big) \prod_{b=1}^{D} \big(\widehat{\varphi}^b - \widehat{\overline{\varphi}}^b\big)\, \mathrm{e}^{-\frac{\beta}{\hbar}\widehat{\mathscr{H}}} \,. \tag{6.1}$$

The chirality operators $\Gamma_{(5)}$ and $\widetilde{\Gamma}_{(5)}$ are given in terms of $\Gamma^a = \sqrt{2}\widehat{\psi}_1^a$ and $\widetilde{\Gamma}^a = \sqrt{2}\widehat{\psi}_2^a$, respectively:

$$\Gamma_{(5)} \equiv (-i)^{D/2}\Gamma^1\cdots\Gamma^D = (-i)^{D/2}2^{D/2}\widehat{\psi}_1^1\cdots\widehat{\psi}_1^D = (-i)^{D/2}\prod_{a=1}^{D}\big(\widehat{\varphi}^a + \widehat{\overline{\varphi}}^a\big)\,, \tag{6.2a}$$

$$\widetilde{\Gamma}_{(5)} \equiv (-i)^{D/2}\widetilde{\Gamma}^1\cdots\widetilde{\Gamma}^D = (-i)^{D/2}2^{D/2}\widehat{\psi}_2^1\cdots\widehat{\psi}_2^D = (-i)^{D/2}(-i)^D\prod_{a=1}^{D}\big(\widehat{\varphi}^a - \widehat{\overline{\varphi}}^a\big)\,. \tag{6.2b}$$

Notice that since the non-trivial values are given when D is even number, we find $(-i)^{2D} = 1$. Then we formulate the Euler characteristic in terms of the transition element and effective action (where $x = x_0 + q$):

$$\begin{aligned}
\chi = \lim_{\beta\to 0}\int \mathrm{d}^D x_0 \sqrt{g(x_0)} \prod_{a=1}^{D}\Big(\mathrm{d}\overline{\eta}_a \mathrm{d}\eta^a \mathrm{d}\zeta^a \mathrm{d}\overline{\zeta}^a \mathrm{d}\overline{\lambda}_a \mathrm{d}\lambda^a\Big)\, \mathrm{e}^{\overline{\zeta}\zeta}\,\mathrm{e}^{-\overline{\lambda}\lambda}\,\mathrm{e}^{-\overline{\eta}\eta} \\
\times \langle\,\overline{\zeta}\,|\prod_{b=1}^{D}\big(\widehat{\varphi}^b + \widehat{\overline{\varphi}}^b\big)|\,\lambda\,\rangle\langle\,\overline{\lambda}\,|\prod_{c=1}^{D}\big(\widehat{\varphi}^c - \widehat{\overline{\varphi}}^c\big)|\,\eta\,\rangle\langle\, x_0, \overline{\eta}\,|\exp\Big(-\frac{\beta}{\hbar}\widehat{\mathscr{H}}_H\Big)|\,x_0,\zeta\,\rangle,
\end{aligned} \tag{6.3a}$$

$$\langle\, x_0,\overline{\eta}\,|\exp\Big(-\frac{\beta}{\hbar}\widehat{\mathscr{H}}_H\Big)|\,x_0,\zeta\,\rangle = \frac{1}{(2\pi\beta\hbar)^{D/2}}\mathrm{e}^{\overline{\eta}\zeta}\Big\langle \exp\Big(-\frac{1}{\hbar}S_H^{(\mathrm{int})}\Big)\Big\rangle\,, \tag{6.3b}$$

$$\begin{aligned}
-\frac{1}{\hbar}S_H^{(\mathrm{int})} = &-\frac{1}{\beta\hbar}\int_{-1}^{0}\mathrm{d}\tau\frac{1}{2}\Big[g_{mn}(x) - g_{mn}(x_0)\Big]\Big(\dot{q}^m\dot{q}^n + b^m c^n + a^m a^n\Big) \\
&-\int_{-1}^{0}\mathrm{d}\tau\dot{q}^m\Big(\omega_{mab}(x)\,(\overline{\eta}+\overline{\xi}_{\mathrm{qu}})^a\,(\zeta+\xi_{\mathrm{qu}})^b - \frac{1}{2}H_{mab}(x)\Big\{(\zeta+\xi_{\mathrm{qu}})^{ab} + (\overline{\eta}+\overline{\xi}_{\mathrm{qu}})^{ab}\Big\}\Big) \\
&+\frac{\beta\hbar}{2}\int_{-1}^{0}\mathrm{d}\tau\, R_{cdab}(\omega(x))\,(\zeta+\xi_{\mathrm{qu}})^a(\overline{\eta}+\overline{\xi}_{\mathrm{qu}})^b(\zeta+\xi_{\mathrm{qu}})^c(\overline{\eta}+\overline{\xi}_{\mathrm{qu}})^d \\
&-\frac{\beta\hbar}{8}\int_{-1}^{0}\mathrm{d}\tau H_{abe}(x)H_{cd}{}^{e}(x)\Big\{(\zeta+\xi_{\mathrm{qu}})^{abcd} + (\overline{\eta}+\overline{\xi}_{\mathrm{qu}})^{abcd} - 2(\zeta+\xi_{\mathrm{qu}})^{ab}(\overline{\eta}+\overline{\xi}_{\mathrm{qu}})^{cd}\Big\} \\
&-\frac{\beta\hbar}{6}\int_{-1}^{0}\mathrm{d}\tau\,\partial_m(H_{npq}(x))\Big\{(\overline{\eta}+\overline{\xi}_{\mathrm{qu}})^m(\zeta+\xi_{\mathrm{qu}})^{npq} + (\zeta+\xi_{\mathrm{qu}})^m(\overline{\eta}+\overline{\xi}_{\mathrm{qu}})^{npq}\Big\} \\
&-\frac{\beta\hbar}{8}\int_{-1}^{0}\mathrm{d}\tau\,\mathcal{G}_2(x)\,,
\end{aligned} \tag{6.3c}$$

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where the functional $\mathcal{G}_2(x)$ is given in (3.20d). Now let us analyze fermionic measure in the form (6.3). The effective action $S^{(\text{int})}$ contains ξ^a and $\overline{\xi}^a$ whose boundaries are ζ and $\overline{\eta}$, respectively, and η, $\overline{\zeta}$ and λ, $\overline{\lambda}$ do not appear in $S^{(\text{int})}$. Then let us rewrite the path integral measure with fermions:

$$\begin{aligned}\prod_a \mathrm{d}\overline{\eta}_a \mathrm{d}\eta^a\, \mathrm{d}\zeta^a \mathrm{d}\overline{\zeta}_a\, \mathrm{d}\overline{\lambda}_a \mathrm{d}\lambda^a &= \prod_a \big(\mathrm{d}\overline{\eta}_a \mathrm{d}\zeta^a\big)\big(\mathrm{d}\overline{\lambda}_a \mathrm{d}\eta^a\big)\big(\mathrm{d}\overline{\zeta}_a \mathrm{d}\lambda^a\big) \\ &= \prod_a \big(\mathrm{d}\overline{\eta}_a \mathrm{d}\zeta^a\big)\Big(2^D \mathrm{d}(\overline{\lambda}+\eta)_a \mathrm{d}(\eta-\overline{\lambda})^a\Big)\Big(2^D \mathrm{d}(\overline{\zeta}+\lambda)_a \mathrm{d}(\lambda-\overline{\zeta})^a\Big)\,, \end{aligned}\tag{6.4a}$$

where we implicitly used the orderings of $\mathrm{d}\overline{\eta}$ and $\mathrm{d}\zeta$ (3.5). Under the integral with $\prod_b(\lambda^b+\overline{\zeta}^b)\prod_c(\eta^b-\overline{\lambda}^b)$ which can be regarded as the fermionic delta functions, we can see $\overline{\zeta}=-\lambda$ and $\eta=\overline{\lambda}$. Then, after a tedious computation, we obtain

$$\int \prod_a \mathrm{d}\overline{\eta}_a \mathrm{d}\eta^a\, \mathrm{d}\zeta^a \mathrm{d}\overline{\zeta}_a\, \mathrm{d}\overline{\lambda}_a \mathrm{d}\lambda^a\, \mathrm{e}^{\overline{\zeta}\zeta-\overline{\lambda}\lambda-\overline{\eta}\eta+\overline{\zeta}\lambda+\overline{\lambda}\eta+\overline{\eta}\zeta} \prod_b \big(\lambda^b+\overline{\zeta}^b\big) \prod_c \big(\eta^b-\overline{\lambda}^b\big) = \int \prod_a \mathrm{d}\overline{\eta}_a \mathrm{d}\zeta^a\,, \tag{6.4b}$$

where we used the fermionic delta functions:

$$\int \prod_a \mathrm{d}(\overline{\zeta}+\eta)_a \prod_b (\eta^b+\overline{\zeta}^b) = 1\,, \qquad (-1)^D \int \prod_{a=1}^D \mathrm{d}\overline{\zeta}\, \mathrm{e}^{-\overline{\zeta}(\eta-\zeta)} = \prod_{a=1}^D \big(\eta^a-\zeta^a\big)\,. \tag{6.4c}$$

Then we rescale the fermion to remove the $\beta\hbar$ dependence on the measure in such a way as

$$\frac{1}{(\beta\hbar)^{D/2}} \prod_{a=1}^D \mathrm{d}\overline{\eta}_a \mathrm{d}\zeta^a \equiv \prod_{a=1}^D \mathrm{d}\overline{\eta}'_a \mathrm{d}\zeta'^a\,, \quad \xi^a \equiv (\beta\hbar)^{-\frac{1}{4}} \xi'^a\,. \tag{6.5}$$

Then the rescaled $S^{(\text{int})}$ is given by (where we omit the prime symbol)

$$\begin{aligned} -\frac{1}{\hbar} S^{(\text{int})} =& -\frac{1}{\beta\hbar} \int_{-1}^0 \mathrm{d}\tau\, \frac{1}{2} \Big[g_{mn}(x) - g_{mn}(x_0)\Big] \Big(\dot{q}^m \dot{q}^n + b^m c^n + a^m a^n\Big) \\ & - \frac{1}{\sqrt{\beta\hbar}} \int_{-1}^0 \mathrm{d}\tau \dot{q}^m \Big(\omega_{mab}(x)(\overline{\eta}+\overline{\xi}_{\text{qu}})^a(\zeta+\xi_{\text{qu}})^b - \frac{1}{2} H_{mab}(x) \Big\{(\zeta+\xi_{\text{qu}})^{ab} + (\overline{\eta}+\overline{\xi}_{\text{qu}})^{ab}\Big\}\Big) \\ & + \frac{1}{2} \int_{-1}^0 \mathrm{d}\tau\, R_{cdab}(\omega(x))\, (\zeta+\xi_{\text{qu}})^a (\overline{\eta}+\overline{\xi}_{\text{qu}})^b (\zeta+\xi_{\text{qu}})^c (\overline{\eta}+\overline{\xi}_{\text{qu}})^d \\ & - \frac{1}{8} \int_{-1}^0 \mathrm{d}\tau H_{abe}(x) H_{cd}{}^e(x) \Big\{(\zeta+\xi_{\text{qu}})^{abcd} + (\overline{\eta}+\overline{\xi}_{\text{qu}})^{abcd} - 2(\zeta+\xi_{\text{qu}})^{ab}(\overline{\eta}+\overline{\xi}_{\text{qu}})^{cd}\Big\} \\ & - \frac{1}{6} \int_{-1}^0 \mathrm{d}\tau\, \partial_m(H_{npq}(x)) \Big\{(\overline{\eta}+\overline{\xi}_{\text{qu}})^m (\zeta+\xi_{\text{qu}})^{npq} + (\zeta+\xi_{\text{qu}})^m (\overline{\eta}+\overline{\xi}_{\text{qu}})^{npq}\Big\} \\ & - \frac{\beta\hbar}{8} \int_{-1}^0 \mathrm{d}\tau\, \mathcal{G}_2(x)\,. \end{aligned}\tag{6.6}$$

Notice that the bosonic and fermionic propagators are now proportional to $\beta\hbar$ and $\sqrt{\beta\hbar}$, respectively (we have also rescaled the fermion propagator):

$$\big\langle q^m(\sigma) q^n(\tau) \big\rangle = -\beta\hbar\, g^{mn}(z)\, \Delta(\sigma,\tau)\,, \tag{6.7a}$$

$$\big\langle \xi^a_{\text{qu}}(\sigma) \overline{\xi}^b_{\text{qu}}(\tau) \big\rangle = \sqrt{\beta\hbar}\, \delta^{ab}\, \theta(\sigma-\tau)\,. \tag{6.7b}$$

Then we easily find that each contraction among quantum fields yields Feynman graphs of higher order in $\beta\hbar$, which goes to zero in the limit $\beta\hbar \to 0$. Only the interaction terms given by background fields x_0^m, ζ^a and $\overline{\eta}^a$ are independent of β and they give rise to the relevant Feynman graphs. Then, we can truncate $S^{(\text{int})}$ in order to obtain the Euler characteristics on the D-dimensional geometry $\mathcal{M}$ in the path integral formalism:

$$\chi(\mathcal{M}) = \frac{1}{(2\pi)^{D/2}} \int_{\mathcal{M}} \mathrm{d}^D x_0 \sqrt{g(x_0)} \prod_{a=1}^{D} \mathrm{d}\overline{\eta}_a \mathrm{d}\zeta^a \left\langle \mathrm{e}^{-\frac{1}{\hbar} S^{(\text{int})}} \right\rangle , \tag{6.8a}$$

$$-\frac{1}{\hbar} S^{(\text{int})} = -\frac{1}{4} R_{abcd}(\omega(x_0))\, \zeta^{ab} \overline{\eta}^{cd} - \frac{1}{6} \partial_a (H_{bcd})(x_0) \Big(\overline{\eta}^a \zeta^{bcd} + \zeta^a \overline{\eta}^{bcd} \Big) , \tag{6.8b}$$

where we used $H_{abc}(x_0) = 0$, $R_{cdab}(\omega) = R_{abcd}(\omega)$ and the second Bianchi identity $R_{abcd}(\omega) + R_{acdb}(\omega) + R_{adbc}(\omega) = 0$ without torsion: $R_{abcd}(\omega)\, \zeta^a \overline{\eta}^b \zeta^c \overline{\eta}^d = -\frac{1}{2} R_{abcd}(\omega)\, \zeta^{ab} \overline{\eta}^{cd}$. Since there exist only background fields, we do not have to introduce quantum propagators to contract interaction terms. The Feynman amplitude of the path integral is given only by the expansion of $\exp(-\frac{1}{\hbar} S^{(\text{int})})$ with noticing that the number of ζ should be equal to the number of $\overline{\eta}$ to saturate the fermionic path integral measure. Since each term in (6.8b) carries even number of background fermions ζ and $\overline{\eta}$, the path integral with $D = 2n+1$ becomes trivial.

Next let us investigate the formulation in various geometries in diverse dimensions. We can easily find that the second and the third terms do not contribute to the Feynman graphs in the case of $D = 2$. This is consistent with the fact there does not exist a totally antisymmetric torsion in two-dimensional geometry.

6.2 Euler characteristics

Next let us investigate the formulation in various geometries in diverse dimensions. We can easily find that the second and the third terms do not contribute to the Feynman graphs in the case of $D = 2$. This is consistent with the fact there does not exist a totally antisymmetric torsion in two-dimensional geometry.

6.2.1 Riemannian manifold

It is worth reviewing the case of the Riemannian manifold without torsion. The action is given as

$$-\frac{1}{\hbar} S^{(\text{int})} = -\frac{1}{4} R_{abcd}(\omega)\, \zeta^{ab} \overline{\eta}^{cd} , \tag{6.9}$$

Then the path integral formulation is described in the following way:

$$\begin{aligned}
\chi(\mathcal{M}) &= \frac{1}{(2\pi)^{D/2}} \int_{\mathcal{M}} \mathrm{d}^D x_0 \sqrt{g(x_0)} \prod_{a=1}^{D} \mathrm{d}\overline{\eta}^a \mathrm{d}\zeta^a \, \exp\Big(-\frac{1}{4} R_{abcd}(\omega)\, \zeta^{ab} \overline{\eta}^{cd} \Big) \\
&= \frac{1}{(8\pi)^n n!}\, \mathcal{E}_{a_1 \cdots a_{2n}} \mathcal{E}^{b_1 \cdots b_{2n}} \int_{\mathcal{M}} \mathrm{d}^{2n} x_0 \sqrt{g(x_0)} \Big(R^{a_1 a_2}{}_{b_1 b_2}(\omega) \cdots R^{a_{2n-1} a_{2n}}{}_{b_{2n-1} b_{2n}}(\omega) \Big) \\
&= \frac{1}{(4\pi)^n n!}\, \mathcal{E}_{a_1 \cdots a_{2n}} \int_{\mathcal{M}} R^{a_1 a_2}(\omega) \wedge \cdots \wedge R^{a_{2n-1} a_{2n}}(\omega) ,
\end{aligned} \tag{6.10}$$

where we used the formulae in Euclidean space:

$$\mathrm{d}^{2n}x_0\sqrt{g(x_0)}\,\mathcal{E}^{b_1\cdots b_{2n}} = e^{b_1}\wedge\cdots\wedge e^{b_{2n}}\ , \quad R^{ab}(\omega) = \frac{1}{2}R^{ab}{}_{cd}(\omega)\,e^c\wedge e^d\ . \tag{6.11}$$

Non-trivial value of $\chi(\mathcal{M})$ is given only when $D = 2n = 2k$ and all indices of totally antisymmetric tensor $\mathcal{E}_{abcd\cdots}$ are the frame (local Lorentz) indices with Euclidean signature. Then we do not mind the positions of the indices.[3] We also used the following formulae in the same way as (4.14):

$$\int \mathrm{d}\zeta_1\cdots\mathrm{d}\zeta_{2n}\,\zeta_{1\cdots 2n} = (-1)^n\ , \quad \int \mathrm{d}\overline{\eta}^{2n}\cdots\mathrm{d}\overline{\eta}^1\,\overline{\eta}^{12\cdots 2n} = 1\ . \tag{6.12}$$

6.2.2 Torsional manifold

In this case we should analyze the full action in (6.8b):

$$-\frac{1}{\hbar}S^{(\mathrm{int})} = -\frac{1}{4}R_{abcd}(\omega)\,\zeta^{ab}\overline{\eta}^{cd} - \frac{1}{6}\partial_a(H_{bcd})\Big(\overline{\eta}^a\zeta^{bcd} + \zeta^a\overline{\eta}^{bcd}\Big)\ . \tag{6.13}$$

We omitted the argument x_0. In the same as the analysis on the Riemannian manifold, we can only investigate the case $D = 2n$, i.e., the case of the even-dimensional manifolds. The expectation value of the exponent is

$$\begin{aligned}\big\langle \mathrm{e}^{-\frac{1}{\hbar}S^{(\mathrm{int})}}\big\rangle &= \exp\bigg(-\frac{1}{4}R_{abcd}(\omega)\,\zeta^{ab}\overline{\eta}^{cd} - \frac{1}{6}\partial_a(H_{bcd})\big(\overline{\eta}^a\zeta^{bcd} + \zeta^a\overline{\eta}^{bcd}\big)\bigg) \qquad (6.14)\\ &= \exp\bigg(-\frac{1}{4}R_{abcd}(\omega)\zeta^{ab}\overline{\eta}^{cd}\bigg)\exp\bigg(-\frac{1}{6}\partial_a(H_{bcd})\overline{\eta}^a\zeta^{bcd}\bigg)\exp\bigg(-\frac{1}{6}\partial_a(H_{bcd})\zeta^a\overline{\eta}^{bcd}\bigg).\end{aligned}$$

Since the path integral measure in (6.8) requires that the number of the background fermions ζ should be equal to the number of $\overline{\eta}$, the third exponent in (6.15) should be contracted only with the second exponent. The second and the third exponents cannot be contracted with the first exponent. Then (6.15) is truncated to

$$\begin{aligned}\big\langle \mathrm{e}^{-\frac{1}{\hbar}S^{(\mathrm{int})}}\big\rangle &\sim \sum_{k+2\ell=n}\frac{1}{k!\ell!\ell!}\bigg(-\frac{1}{4}R_{abcd}(\omega)\zeta^{ab}\overline{\eta}^{cd}\bigg)^k\bigg(-\frac{1}{6}\partial_a(H_{bcd})\overline{\eta}^a\zeta^{bcd}\bigg)^\ell\bigg(-\frac{1}{6}\partial_a(H_{bcd})\zeta^a\overline{\eta}^{bcd}\bigg)^\ell \\ &= \sum_{k+2\ell=n}\frac{2^{2\ell}}{3^{2\ell}k!\ell!\ell!}\bigg(-\frac{1}{4}\bigg)^n\Big(R_{a_1a_2b_1b_1}\cdots R_{a_{2k-1}a_{2k}b_{2k-1}b_{2k}}\Big) \\ &\quad\times\Big(\partial_{c_1}(H_{d_1d_2d_3})\cdots\partial_{c_\ell}(H_{d_{3\ell-2}d_{3\ell-1}d_{3\ell}})\Big)\Big(\partial_{e_1}(H_{f_1f_2f_3})\cdots\partial_{e_\ell}(H_{f_{3\ell-2}f_{3\ell-1}f_{3\ell}})\Big) \\ &\quad\times\zeta^{a_1\cdots a_{2k}c_1\cdots c_\ell f_1\cdots f_{3\ell}}\overline{\eta}^{b_1\cdots b_{2k}d_1\cdots d_\ell e_1\cdots e_{3\ell}} \\ &= \sum_{k+2\ell=n}\frac{2^{2\ell}}{3^{2\ell}k!\ell!\ell!}\bigg(-\frac{1}{4}\bigg)^n\mathcal{E}^{a_1\cdots a_{2k}c_1\cdots c_\ell f_1\cdots f_{3\ell}}\mathcal{E}^{b_1\cdots b_{2k}d_1\cdots d_\ell e_1\cdots e_{3\ell}}\,\zeta^{1\cdots 2n}\,\overline{\eta}^{1\cdots 2n} \\ &\quad\times\Big(R_{a_1a_2b_1b_1}\cdots R_{a_{2k-1}a_{2k}b_{2k-1}b_{2k}}\Big)\Big(\partial_{c_1}(H_{d_1d_2d_3})\cdots\partial_{c_\ell}(H_{d_{3\ell-2}d_{3\ell-1}d_{3\ell}})\Big) \\ &\quad\times\Big(\partial_{e_1}(H_{f_1f_2f_3})\cdots\partial_{e_\ell}(H_{f_{3\ell-2}f_{3\ell-1}f_{3\ell}})\Big)\end{aligned} \tag{6.15}$$

[3]In the case of curved indices, the positions of indices are quite important we should really mind whether $\varepsilon_{mnpq\cdots}$ is a tensor or a tensor density. In the case of frame coordinate indices, the weight $\sqrt{g(x_0)}$ does not appear.

Substituting this into (6.8), we obtain

$$\chi(\mathcal{M}) = \frac{1}{(8\pi)^n} \sum_{k+2\ell=n} \frac{2^{2\ell}}{3^{2\ell} k!\ell!\ell!} \mathcal{E}^{a_1\cdots a_{2k} c_1 \cdots c_\ell f_1 \cdots f_{3\ell}} \mathcal{E}^{b_1 \cdots b_{2k} d_1 \cdots d_\ell e_1 \cdots e_{3\ell}}$$
$$\times \int_{\mathcal{M}} \mathrm{d}^{2n} x_0 \sqrt{g(x_0)} \Big(R_{a_1 a_2 b_1 b_1} \cdots R_{a_{2k-1} a_{2k} b_{2k-1} b_{2k}} \Big)$$
$$\times \Big(\partial_{c_1}(H_{d_1 d_2 d_3}) \cdots \partial_{c_\ell}(H_{d_{3\ell-2} d_{3\ell-1} d_{3\ell}}) \Big) \Big(\partial_{e_1}(H_{f_1 f_2 f_3}) \cdots \partial_{e_\ell}(H_{f_{3\ell-2} f_{3\ell-1} f_{3\ell}}) \Big). \tag{6.16}$$

Fortunately, we can furthermore reduce the above representation by using the second Bianchi identity of the Riemann tensor (A.6b) and the closed condition $\mathrm{d}H = 0$. For simplicity, let us analyze the case $k = 1$, $\ell = 2$, from which we can read a general statement:

$$\mathcal{E}^{a_1 a_2 c_1 c_2 f_1 \cdots f_6} \mathcal{E}^{b_1 b_2 d_1 d_2 e_1 \cdots e_6} \int \mathrm{d}^{2n} x_0 \sqrt{g(x_0)} R_{a_1 a_2 b_1 b_1} \partial_{c_1}(H_{d_1 d_2 d_3}) \partial_{c_2}(H_{d_4 d_5 d_6}) \partial_{e_1}(H_{f_1 f_2 f_3}) \times$$
$$\partial_{e_2}(H_{f_4 f_5 f_6}) = \mathcal{E}^{a_1 \cdots f_6} \mathcal{E}^{b_1 \cdots e_6} \int \mathrm{d}^{2n} x_0 \partial_{c_1} \Big(\text{all terms} \Big)$$
$$- \mathcal{E}^{a_1 \cdots f_6} \mathcal{E}^{b_1 \cdots e_6} \int \mathrm{d}^{2n} x_0 \sqrt{g(x_0)} \partial_{c_1} \Big\{ R_{a_1 a_2 b_1 b_1} \Big\} (H_{d_1 d_2 d_3}) \partial_{c_2}(H_{d_4 d_5 d_6}) \partial_{e_1}(H_{f_1 f_2 f_3}) \partial_{e_2}(H_{f_4 f_5 f_6})$$
$$- \mathcal{E}^{a_1 \cdots f_6} \mathcal{E}^{b_1 \cdots e_6} \int \mathrm{d}^{2n} x_0 \sqrt{g(x_0)} R_{a_1 a_2 b_1 b_1} (H_{d_1 d_2 d_3}) \Big\{ \partial_{c_1} \partial_{c_2}(H_{d_4 d_5 d_6}) \Big\} \partial_{e_1}(H_{f_1 f_2 f_3}) \partial_{e_2}(H_{f_4 f_5 f_6})$$
$$- \mathcal{E}^{a_1 \cdots f_6} \mathcal{E}^{b_1 \cdots e_6} \int \mathrm{d}^{2n} x_0 \sqrt{g(x_0)} R_{a_1 a_2 b_1 b_1} (H_{d_1 d_2 d_3}) \partial_{c_2}(H_{d_4 d_5 d_6}) \partial_{c_1} \Big\{ \partial_{e_1}(H_{f_1 f_2 f_3}) \partial_{e_2}(H_{f_4 f_5 f_6}) \Big\}. \tag{6.17}$$

The first term in (6.17) vanishes if there are no boundaries on the manifold. The second term also vanishes via the second Bianchi identity (A.6b). The third term is zero because the derivatives are symmetric, while the indices are anti-symmetric under the existence of $\mathcal{E}^{a_1 \cdots f_6}$. The fourth term also vanishes because the closed condition $\mathrm{d}H = 0$ appears as $\mathcal{E}^{a_1 \cdots f_6} \partial_{c_1}(H_{f_1 f_2 f_3}) = 0$. Other derivatives also yield the same result. Thus we find that the second and the third exponents in (6.15) should not contribute to the Euler characteristics and we can set $\ell = 0$. We conclude that the Euler characteristics on the torsional manifold without boundary is equal to the ones on the Riemannian manifold (6.10):

$$\chi(\mathcal{M}) = \frac{1}{(8\pi)^n n!} \mathcal{E}^{a_1 \cdots a_{2n}} \mathcal{E}^{b_1 \cdots b_{2n}} \int_{\mathcal{M}} \mathrm{d}^{2n} x_0 \sqrt{g(x_0)} \Big(R_{a_1 a_2 b_1 b_1} \cdots R_{a_{2n-1} a_{2n} b_{2n-1} b_{2n}} \Big)$$
$$= \frac{1}{(4\pi)^n n!} \mathcal{E}_{a_1 \cdots a_{2n}} \int_{\mathcal{M}} R^{a_1 a_2}(\omega) \wedge \cdots \wedge R^{a_{2n-1} a_{2n}}(\omega) \,. \tag{6.18}$$

7. Witten index in $\mathcal{N} = 2$ quantum mechanics II

Finally we will discuss the derivation of the Hirzebruch signature on a torsional manifold in the path integral formalism. We also use the $\mathcal{N} = 2$ supersymmetric quantum mechanical path integral, while we only insert $\Gamma_{(5)}$ into the transition element instead of the insertion $\Gamma_{(5)} \widetilde{\Gamma}_{(5)}$ in the case of the Euler characteristics. We review the derivation of the signature on the Riemannian manifold. Next we discuss the analysis of the signature on a torsional manifold in the same strategy.

7.1 Formulation

As mentioned in the introduction, the Hirzebruch signature is a topological invariant which gives the difference between the number of self-dual forms and the number of anti-self-dual forms on the manifold. Since we analyze the difference of the forms, we analyze another Witten index defined in the $\mathcal{N}=2$ supersymmetric quantum mechanics in the following form (see section 14.3 in [19]):

$$\sigma \equiv \lim_{\beta \to 0} \mathrm{Tr}\big\{ \Gamma_{(5)} \mathrm{e}^{-\beta \mathscr{H}} \big\} = \lim_{\beta \to 0} (-i)^{D/2} \mathrm{Tr} \prod_{a=1}^{D} \big(\widehat{\varphi}^a + \widehat{\overline{\varphi}}^a \big) \, \mathrm{e}^{-\frac{\beta}{\hbar} \widehat{\mathscr{H}}} \, . \tag{7.1}$$

Here we did not insert $2^{-D/2}$ because in this system ψ_2^a is also dynamical. The chirality operators $\Gamma_{(5)}$ is again given in terms of the operators $\widehat{\psi}_1^a$:

$$\Gamma_{(5)} \equiv (-i)^{D/2} \Gamma^1 \cdots \Gamma^D = (-i)^{D/2} 2^{D/2} \widehat{\psi}_1^1 \cdots \widehat{\psi}_1^D = (-i)^{D/2} \prod_{a=1}^{D} \big(\widehat{\varphi}^a + \widehat{\overline{\varphi}}^a \big) \, . \tag{7.2}$$

Notice that since the non-trivial values are given when D is even number, we find $(-i)^{2D} = 1$. In addition, we prepare the trace formula and the complete set of the fermion coherent states (3.5). We obtain the explicit expression of the topological invariants with respect to the $\mathcal{N}=2$ quantum mechanical path integral in the same way as (6.3):

$$\begin{aligned} \sigma = \lim_{\beta \to 0} \frac{(-i)^{D/2}}{(2\pi\beta\hbar)^{D/2}} \int \mathrm{d}^D x_0 \sqrt{g(x_0)} \prod_{a=1}^{D} \Big(\mathrm{d}\overline{\eta}_a \mathrm{d}\eta^a \, \mathrm{d}\zeta^a \mathrm{d}\overline{\zeta}_a \Big) & \\ \times \, \mathrm{e}^{\overline{\zeta}\zeta + \overline{\zeta}\eta - \overline{\eta}\eta + \overline{\eta}\zeta} \prod_b \big(\eta^b + \overline{\zeta}^b \big) \Big\langle \exp\Big(- \frac{1}{\hbar} S_H^{(\mathrm{int})} \Big) \Big\rangle & \, , \end{aligned} \tag{7.3}$$

where $S^{(\mathrm{int})}$ in (7.3) is also given by (6.3c) which appeared in the previous subsection. Now let us consider the fermionic measure in this path integral form. In the same way as the Dirac index, we obtain

$$\begin{aligned} & \int \prod_a \mathrm{d}\overline{\eta}_a \mathrm{d}\eta^a \mathrm{d}\zeta^a \mathrm{d}\overline{\zeta}_a \, \mathrm{e}^{\overline{\zeta}\zeta + \overline{\zeta}\eta - \overline{\eta}\eta + \overline{\eta}\zeta} \prod_b (\eta^b + \overline{\zeta}^b) \\ & \quad = (-2)^D \int \prod_a \mathrm{d}\overline{\eta}_a \mathrm{d}\zeta^a \, \mathrm{d}(\overline{\zeta} + \eta)_a \mathrm{d}(\eta - \overline{\zeta})^a \, \mathrm{e}^{-\frac{1}{2}(\eta - \overline{\zeta})(\zeta - \overline{\eta})} \prod_b (\eta^b + \overline{\zeta}^b) \\ & \quad = \int \prod_a \mathrm{d}\overline{\eta}_a \mathrm{d}\zeta^a \prod_b (\zeta^b - \overline{\eta}^b) \, . \end{aligned} \tag{7.4}$$

This measure gives the fermionic delta function which indicates the coincidence of the background fermions $\zeta^a = \overline{\eta}^a$:

$$\int \prod_a \mathrm{d}\overline{\eta}_a \prod_b (\zeta^b - \overline{\eta}^b) \, f(\overline{\eta}) = f(\zeta) \, . \tag{7.5}$$

To remove the β dependence in the path integral measure, we rescale the fermion

$$\frac{1}{(\beta\hbar)^{D/2}}\int\prod_a \mathrm{d}\overline{\eta}_a \mathrm{d}\zeta^a \prod_b(\zeta^b-\overline{\eta}^b) \equiv \int\prod_a \mathrm{d}\overline{\eta}'_a \mathrm{d}\zeta'^a \prod_b(\zeta'^b-\overline{\eta}'^b) \,, \tag{7.6a}$$

$$\overline{\eta}^a \equiv \Big(\frac{1}{\beta\hbar}\Big)^{1/2}\overline{\eta}'^a \,, \quad \zeta^a \equiv \Big(\frac{1}{\beta\hbar}\Big)^{1/2}\zeta'^a \,. \tag{7.6b}$$

Then the rescaled $S^{(\mathrm{int})}$ (3.20c) in the path integral is given by (where we omit the prime symbol)

$$\sigma = \lim_{\beta\to 0}\frac{(-i)^{D/2}}{(2\pi)^{D/2}}\int \mathrm{d}^D x_0\sqrt{g(x_0)}\prod_{a=1}^{D}\mathrm{d}\overline{\eta}_a\mathrm{d}\zeta^a\prod_{b=1}^{D}(\zeta^b-\overline{\eta}^b)\Big\langle \exp\Big(-\frac{1}{\hbar}S_H^{(\mathrm{int})}\Big)\Big\rangle \,, \tag{7.7a}$$

$$\begin{aligned}
-\frac{1}{\hbar}S_H^{(\mathrm{int})} = & -\frac{1}{\beta\hbar}\int_{-1}^{0}\mathrm{d}\tau\frac{1}{2}\Big[g_{mn}(x)-g_{mn}(x_0)\Big]\Big(\dot{q}^m\dot{q}^n+b^m c^n+a^m a^n\Big) \\
& -\frac{1}{\beta\hbar}\int_{-1}^{0}\mathrm{d}\tau\dot{q}^m\Big(\omega_{mab}(x)\big(\overline{\eta}+\overline{\xi}_{\mathrm{qu}}\big)^a\big(\zeta+\xi_{\mathrm{qu}}\big)^b-\frac{1}{2}H_{mab}(x)\Big\{\big(\zeta+\xi_{\mathrm{qu}}\big)^{ab}+\big(\overline{\eta}+\overline{\xi}_{\mathrm{qu}}\big)^{ab}\Big\}\Big) \\
& +\frac{1}{2\beta\hbar}\int_{-1}^{0}\mathrm{d}\tau\,R_{cdab}(\omega(x))\,(\zeta+\xi_{\mathrm{qu}})^a(\overline{\eta}+\overline{\xi}_{\mathrm{qu}})^b(\zeta+\xi_{\mathrm{qu}})^c(\overline{\eta}+\overline{\xi}_{\mathrm{qu}})^d \\
& -\frac{1}{8\beta\hbar}\int_{-1}^{0}\mathrm{d}\tau H_{abe}H_{cd}{}^{e}(x)\Big\{(\zeta+\xi_{\mathrm{qu}})^{abcd}+(\overline{\eta}+\overline{\xi}_{\mathrm{qu}})^{abcd}-2(\zeta+\xi_{\mathrm{qu}})^{ab}(\overline{\eta}+\overline{\xi}_{\mathrm{qu}})^{cd}\Big\} \\
& -\frac{1}{6\beta\hbar}\int_{-1}^{0}\mathrm{d}\tau\,\partial_m(H_{npq})(x)\Big\{(\overline{\eta}+\overline{\xi}_{\mathrm{qu}})^m(\zeta+\xi_{\mathrm{qu}})^{npq}+(\zeta+\xi_{\mathrm{qu}})^m(\overline{\eta}+\overline{\xi}_{\mathrm{qu}})^{npq}\Big\} \\
& -\frac{\beta\hbar}{8}\int_{-1}^{0}\mathrm{d}\tau\,\mathcal{G}_2(x) \,.
\end{aligned} \tag{7.7b}$$

The bosonic and fermionic propagators are of order in $\beta\hbar$. Let us truncate this action. In the same analogy to the Dirac index, disconnected Feynman graphs might contribute to the amplitude. In the same way as previous case, the fermion propagator is given by

$$\big\langle \xi^a_{\mathrm{qu}}(\sigma)\overline{\xi}^b_{\mathrm{qu}}(\tau)\big\rangle = \beta\hbar\,\delta^{ab}\theta(\sigma-\tau) \,. \tag{7.8}$$

7.2 Hirzebruch signature

7.2.1 Riemannian manifold

This case is quite simple. Since there are no background interaction terms of order in $(\beta\hbar)^{-1}$ which contribute to the disconnected graphs, we only consider one-loop Feynman graphs. Then, we neglect interaction terms carrying more than three quantum fields. We can also neglect the last line in (7.7b) which yields the graphs of higher order in $\beta\hbar$. We also use the condition by Riemann normal coordinate frame $\partial_p g_{mn}(x_0)=\omega_{mab}(x_0)=0$ at the point x_0. We can further neglect interaction terms which are irrelevant in the vanishing limit $\beta\to 0$. By using the Riemann normal coordinates on the second line in (7.7b), the fermionic delta function (7.4) and the first Bianchi identity (A.6a) acting on the fourth line in (7.7b), we obtain a much simpler expression of the Hirzebruch signature:

$$\sigma = \lim_{\beta \to 0} \frac{(-i)^{D/2}}{(2\pi)^{D/2}} \int \mathrm{d}^D x_0 \sqrt{g(x_0)} \prod_{a=1}^{D} \mathrm{d}\zeta^a \Big\langle \exp\Big(- \frac{1}{\hbar} S^{(\mathrm{int})} \Big) \Big\rangle \, , \tag{7.9a}$$

$$\begin{aligned} -\frac{1}{\hbar} S^{(\mathrm{int})} &= -\frac{1}{2\beta\hbar} R_{mnab}(\omega(x_0))\, \zeta^{ab} \int_{-1}^{0} \mathrm{d}\tau\, q^m \dot{q}^n \\ &\quad + \frac{1}{2\beta\hbar} R_{abcd}(\omega(x_0))\, \zeta^{ab} \Big(-\frac{1}{2} \int_{-1}^{0} \mathrm{d}\tau \xi^c_{\mathrm{qu}} \xi^d_{\mathrm{qu}} - \frac{1}{2} \int_{-1}^{0} \mathrm{d}\tau \overline{\xi}^c_{\mathrm{qu}} \overline{\xi}^d_{\mathrm{qu}} + \int_{-1}^{0} \mathrm{d}\tau \xi^c_{\mathrm{qu}} \overline{\xi}^d_{\mathrm{qu}} \Big) . \end{aligned} \tag{7.9b}$$

We should notice that the fermionic fields in the above path integral have *anti-periodic* boundary condition. Originally the fermionic fields are introduced as the fields with anti-periodic boundary condition (see the discussion in section 2.4 of [26]), which is changed by the insertion of operators. Now, in the form (7.9) there are no additional operator insertions in the path integral measure. Thus the fermions in (7.9) keep the anti-periodic boundary condition.

We can easily find that the Feynman graphs will be described as the trace of Riemann curvature two-form in the same way as the Pontrjagin classes. Here let us remember a property that the trace of odd number of Riemann curvature two-form vanishes $\mathrm{tr}(R^{2k-1}) = 0$. On the other hand, the Feynman one-loop graph which contains all of three interaction terms in the second line in (7.9b) always has odd number of the interaction vertices. This indicates that the third interaction term in the second line $\xi^c_{\mathrm{qu}} \overline{\xi}^d_{\mathrm{qu}}$ should not be connected to the other two interactions ($\xi^c_{\mathrm{qu}} \xi^d_{\mathrm{qu}}$ and $\overline{\xi}^c_{\mathrm{qu}} \overline{\xi}^d_{\mathrm{qu}}$) in the graphs. These other two terms should be connected to each other. Furthermore, because of the anti-periodicity of the fermions, we also find that the closed loop graphs which contain only the third interaction $\xi^c_{\mathrm{qu}} \overline{\xi}^d_{\mathrm{qu}}$ vanish in the same reason as the vanishing closed loop graphs of $\hat{c}$-ghost in (5.11b), which also has the anti-periodic boundary condition. The term in the first line exactly gives a same Feynman graphs as the Pontrjagin classes (4.15). Summarizing these comments, here let us again describe the action in (7.9):

$$\begin{aligned} -\frac{1}{\hbar} S^{(\mathrm{int})} &= -\frac{1}{\beta\hbar} R_{mn} \int_{-1}^{0} \mathrm{d}\tau\, q^m \dot{q}^n - \frac{1}{2\beta\hbar} R_{cd} \int_{-1}^{0} \mathrm{d}\tau\, \xi^c_{\mathrm{qu}} \xi^d_{\mathrm{qu}} - \frac{1}{2\beta\hbar} R_{cd} \int_{-1}^{0} \mathrm{d}\tau\, \overline{\xi}^c_{\mathrm{qu}} \overline{\xi}^d_{\mathrm{qu}} \\ &\equiv -\frac{1}{\hbar} \mathcal{S}_{\mathrm{p}} - \frac{1}{\hbar} \mathcal{S} - \frac{1}{\hbar} \overline{\mathcal{S}} \, , \end{aligned} \tag{7.10a}$$

$$R_{cd} \equiv \frac{1}{2} R_{cdab}(\omega(x_0))\, \zeta^{ab} = \frac{1}{2} R_{abcd}(\omega(x_0))\, \zeta^{ab} \, . \tag{7.10b}$$

Let us rewrite the exponent $\langle \exp(-\frac{1}{\hbar} S^{(\mathrm{int})}) \rangle$ in terms of the effective action W in such a way as

$$\begin{aligned} -\frac{1}{\hbar} W &= \log \Big\langle \exp\Big(-\frac{1}{\hbar} S^{(\mathrm{int})} \Big) \Big\rangle = \sum_{k=1}^{\infty} \frac{1}{k!} \Big\langle\!\!\Big\langle \Big(-\frac{1}{\hbar} S^{(\mathrm{int})} \Big)^k \Big\rangle\!\!\Big\rangle \\ &\sim \sum_{k=1}^{\infty} \frac{1}{k!} \Big\langle\!\!\Big\langle \Big(-\frac{1}{\hbar} \mathcal{S}_{\mathrm{p}} \Big)^k \Big\rangle\!\!\Big\rangle + \sum_{k=0}^{\infty} \frac{1}{k!} \frac{k!}{(k/2)!(k/2)!} \Big\langle\!\!\Big\langle \Big(-\frac{1}{\hbar} \mathcal{S} \Big)^{k/2} \Big(-\frac{1}{\hbar} \overline{\mathcal{S}} \Big)^{k/2} \Big\rangle\!\!\Big\rangle \end{aligned}$$

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$$
\begin{aligned}
&= \sum_{k=1}^{\infty} \frac{1}{k!}\big(-\frac{1}{\beta\hbar}\big)^k R_{m_1n_1}\cdots R_{m_kn_k} \int_{-1}^{0} \mathrm{d}\tau_1\cdots\mathrm{d}\tau_k \Big\langle\!\!\Big\langle (q^{m_1}\dot{q}^{n_1})(\tau_1)\cdots(q^{m_k}\dot{q}^{n_k})(\tau_k)\Big\rangle\!\!\Big\rangle \\
&\quad + \sum_{\ell=1}^{\infty} \frac{1}{\ell!\ell!}\Big(-\frac{1}{2\beta\hbar}\Big)^{2\ell} R_{a_1b_1}\cdots R_{a_\ell b_\ell} R_{c_1d_1}\cdots R_{c_\ell d_\ell} \int_{-1}^{0}\mathrm{d}\tau_1\cdots\mathrm{d}\tau_\ell \int_{-1}^{0}\mathrm{d}\sigma_1\cdots\mathrm{d}\sigma_\ell \\
&\quad\quad \times \Big\langle\!\!\Big\langle \big(\xi_{\mathrm{qu}}^{a_1}\xi_{\mathrm{qu}}^{b_1}\big)(\tau_1)\cdots\big(\xi_{\mathrm{qu}}^{a_\ell}\xi_{\mathrm{qu}}^{b_\ell}\big)(\tau_\ell)\big(\overline{\xi}_{\mathrm{qu}}^{c_1}\overline{\xi}_{\mathrm{qu}}^{d_1}\big)(\sigma_1)\cdots\big(\overline{\xi}_{\mathrm{qu}}^{c_\ell}\overline{\xi}_{\mathrm{qu}}^{d_\ell}\big)(\sigma_\ell)\Big\rangle\!\!\Big\rangle , \qquad (7.11)
\end{aligned}
$$

where we extracted terms which contribute to the Feynman graphs in the vanishing limit $\beta \to 0$. The bracket $\langle\!\langle\cdots\rangle\!\rangle$ gives connected Feynman graphs. The number of the vertices $\mathcal{S}$ should be equal to the number of the vertices $\overline{\mathcal{S}}$. Because of this, we find that k should be even: $k = 2\ell$.

Since we have already analyzed the first connected graphs in the Pontrjagin classes (4.18), it is easy to analyze the first term in (7.11):

$$
\begin{aligned}
(\text{1st term}) &= \sum_{k=1}^{\infty} \frac{1}{k!}\Big(-\frac{1}{\beta\hbar}\Big)^k (k-1)!\, 2^{k-1}\big(-\beta\hbar\big)^k \cdot R_{m_1n_1}R_{m_2n_2}\cdots R_{m_kn_k} g^{n_1m_2}g^{n_2m_3}\cdots g^{n_km_1} \\
&\quad \times \int_{-1}^{0}\mathrm{d}\tau_1\cdots\mathrm{d}\tau_k\, \partial_{\tau_1}\Delta(\tau_1,\tau_2)\partial_{\tau_2}\Delta(\tau_2,\tau_3)\cdots\partial_{\tau_{k-1}}\Delta(\tau_{k-1},\tau_k)\partial_{\tau_k}\Delta(\tau_k,\tau_1) \\
&\equiv \frac{1}{2}\sum_{k=2}^{\infty}\frac{1}{k}\mathrm{tr}\big\{(2R)^k\big\} I_k = \frac{1}{2}\mathrm{tr}\log\Big(\frac{R}{\sinh R}\Big) . \qquad (7.12a)
\end{aligned}
$$

Next let us here evaluate the second connected graphs in (7.11). In order to make one-loop graphs, ℓ vertices $\mathcal{S}$ and ℓ vertices $\overline{\mathcal{S}}$ should be alternatively located on the one-loop graph in $(\ell-1)!\ell!$ ways. Furthermore, there are $2^{2\ell-1}$ ways to contract these vertices in terms of fermion propagator (7.8) to yield the trace of curvature two-forms $\mathrm{tr}(R^{2\ell})$ with sign $(-1)^{\ell+1}$, which comes from permutation of indices. Then, the effective action (7.11) is evaluated in the following way:

$$
\begin{aligned}
(\text{2nd term}) &= \sum_{\ell=1}^{\infty}\frac{1}{\ell!\ell!}\Big(-\frac{1}{2\beta\hbar}\Big)^{2\ell}(\ell-1)!\ell!\, 2^{2\ell-1}\,(-1)^{\ell+1}\,\big(\beta\hbar\big)^{2\ell}\cdot\mathrm{tr}\big(R^{2\ell}\big) \\
&\quad \times \int_{-1}^{0}\prod_{i=1}^{\ell}\mathrm{d}\tau_i\mathrm{d}\sigma_i\theta(\tau_1-\sigma_1)\theta(\tau_1-\sigma_\ell)\theta(\tau_2-\sigma_2)\theta(\tau_2-\sigma_1)\cdots\theta(\tau_\ell-\sigma_\ell)\theta(\tau_\ell-\sigma_{\ell-1}) \\
&\equiv \frac{1}{2}\sum_{\ell=1}^{\infty}\frac{(-1)^{\ell+1}}{\ell}\,\mathrm{tr}\big(R^{2\ell}\big)J_{2\ell} = \frac{1}{2}\mathrm{tr}\log\Big(\cosh R\Big) . \qquad (7.13)
\end{aligned}
$$

The term $\ell = 0$ does not contribute to connected graphs because this term does not carry any background fermions. The function $J_{2\ell}$ is defined in such a way as

$$
J_{2\ell} \equiv \int_{-1}^{0}\prod_{i=1}^{\ell}\mathrm{d}\tau_i\,\mathrm{d}\sigma_i\,\theta(\tau_1-\sigma_1)\theta(\tau_1-\sigma_\ell)\,\theta(\tau_2-\sigma_2)\theta(\tau_2-\sigma_1)\cdots\theta(\tau_\ell-\sigma_\ell)\theta(\tau_\ell-\sigma_{\ell-1}) . \qquad (7.14)
$$

Thus, substituting (7.12a) and (7.13) into (7.11), we obtain

$$
-\frac{1}{\hbar}W = \frac{1}{2}\mathrm{tr}\log\Big(\frac{R}{\sinh R}\Big) + \frac{1}{2}\mathrm{tr}\log\Big(\cosh R\Big) = \frac{1}{2}\mathrm{tr}\log\Big(\frac{R}{\tanh R}\Big) . \qquad (7.15)
$$

Rescaling $\zeta^a \to \sqrt{\frac{-i}{2\pi}}\zeta^a$, we finally obtain the Hirzebruch signature on the Riemannian manifold

$$\sigma = \int d^D x_0 \sqrt{g(x_0)} \prod_{a=1}^{D} d\zeta^a \, \exp\left[\frac{1}{2}\mathrm{tr}\log\left(\frac{-iR/2\pi}{\tanh(-iR/2\pi)}\right)\right] , \tag{7.16}$$

or, if we integrate out the fermionic fields and using the following formula (in the same way as (6.12)), we simplify (7.16) and obtain

$$\sigma = \int_{\mathcal{M}} \exp\left[\frac{1}{2}\mathrm{tr}\log\left(\frac{iR/2\pi}{\tanh(iR/2\pi)}\right)\right] , \quad R_{mn} = \frac{1}{2}R_{mnab}(\omega)\, e^a \wedge e^b . \tag{7.17}$$

7.2.2 Torsional manifold

Now let us analyze the signature on the torsional geometry. It seems that the action (7.7b) carries the background interaction terms of order $(\beta\hbar)^{-1}$ in (7.7b), which cause the divergence of the amplitude in the vanishing limit $\beta \to 0$. Fortunately, however, the fermionic delta function (7.4) removes this difficulty:

$$R_{cdab}(\omega(x_0))\Big\{\zeta^a\overline{\eta}^b\zeta^c\overline{\eta}^d\Big\}\Big|_{(7.4)} = R_{[abcd]}(\omega(x_0))\,\zeta^{abcd} = 0 , \tag{7.18a}$$

$$H^e{}_{ab}H_{cde}(x_0)\Big\{\zeta^{abcd} + \overline{\eta}^{abcd} - 2\zeta^{ab}\overline{\eta}^{cd}\Big\}\Big|_{(7.4)} = 2H^e{}_{ab}H_{cde}(x_0)\Big\{\zeta^{abcd} - \zeta^{abcd}\Big\} = 0 , \tag{7.18b}$$

$$\partial_m(H_{npq})(x_0)\Big\{\overline{\eta}^m\zeta^{npq} + \zeta^m\overline{\eta}^{npq}\Big\}\Big|_{(7.4)} = \frac{1}{2}(dH)_{mnpq}(x_0)\,\zeta^{mnpq} = 0 , \tag{7.18c}$$

where we used the first Bianchi identity (A.6a) and the closed condition $dH = 0$. Since we find that there are no background interaction terms with $(\beta\hbar)^{-1}$, it is sufficient to investigate the interaction terms equipped with two quantum fields in order to generate the closed one-loop Feynman graphs. Here let us study the truncation of the action in (7.7b) with the fermionic delta function (7.4). The first and the last lines in (7.7b) disappear. The second and the third lines in (7.7b) are truncated to

$$\begin{aligned}
-\frac{1}{\beta\hbar}\int_{-1}^{0} d\tau \dot{q}^m & \left(\omega_{mab}(x)\big(\overline{\eta}+\overline{\xi}_{\mathrm{qu}}\big)^a\big(\zeta+\xi_{\mathrm{qu}}\big)^b - \frac{1}{2}H_{mab}(x)\Big\{\big(\zeta+\xi_{\mathrm{qu}}\big)^{ab} + \big(\overline{\eta}+\overline{\xi}_{\mathrm{qu}}\big)^{ab}\Big\}\right)\Bigg|_{(7.4)} \\
&= -\frac{1}{2\beta\hbar}R_{abmn}(\omega_-(x_0))\,\zeta^{ab}\int_{-1}^{0} d\tau\, q^m\dot{q}^n \\
&= -\frac{1}{2\beta\hbar}R_{mnab}(\omega_+(x_0))\,\zeta^{ab}\int_{-1}^{0} d\tau\, q^m\dot{q}^n ,
\end{aligned} \tag{7.19a}$$

$$\begin{aligned}
\frac{1}{2\beta\hbar}\int_{-1}^{0} d\tau\, & R_{cdab}(\omega(x))\,(\zeta+\xi_{\mathrm{qu}})^a(\overline{\eta}+\overline{\xi}_{\mathrm{qu}})^b(\zeta+\xi_{\mathrm{qu}})^c(\overline{\eta}+\overline{\xi}_{\mathrm{qu}})^d\Bigg|_{(7.4)} \\
&= \frac{1}{2\beta\hbar}R_{abcd}(\omega(x_0))\,\zeta^{ab}\left(-\frac{1}{2}\int_{-1}^{0} d\tau\,\xi_{\mathrm{qu}}^{cd} - \frac{1}{2}\int_{-1}^{0} d\tau\,\overline{\xi}_{\mathrm{qu}}^{cd} + \int_{-1}^{0} d\tau\,\xi_{\mathrm{qu}}^c\overline{\xi}_{\mathrm{qu}}^d\right) .
\end{aligned} \tag{7.19b}$$

The fourth line in (7.7b) is also truncated to

$$\text{(fouth line)}\Big|_{(7.4)} \sim -\frac{1}{16\beta\hbar}\partial_m\partial_n(H_{abe}H_{cd}{}^e(x_0))\int_{-1}^0 \mathrm{d}\tau\, q^m q^n\Big\{\zeta^{abcd}+\zeta^{abcd}-2\zeta^{ab}\zeta^{cd}\Big\} = 0\,. \tag{7.20}$$

where we used $H_{mnp}(x_0) = 0$ and $\partial_m H_{abc}(x_0) \neq 0$. The fifth line in (7.7b) is more complicated:

$$\begin{aligned}
\text{(fifth line)}\Big|_{(7.4)} &\sim -\frac{1}{6\beta\hbar}\int_{-1}^0 \mathrm{d}\tau\,\bigg(q^r\partial_r\partial_a(H_{bcd}(x_0))\Big\{\overline{\xi}{}^a_{\rm qu}\zeta^{bcd}+3\zeta^{abc}\xi^d_{\rm qu}+\xi^a_{\rm qu}\zeta^{bcd}+3\zeta^{abc}\overline{\xi}{}^d_{\rm qu}\Big\} \\
&\qquad + 3\partial_a(H_{bcd}(x_0))\Big\{\overline{\xi}{}^a_{\rm qu}\xi^b_{\rm qu}\,\zeta^{cd}+\zeta^{ab}\xi^{cd}_{\rm qu}+\xi^a_{\rm qu}\overline{\xi}{}^b_{\rm qu}\,\zeta^{cd}+\zeta^{ab}\overline{\xi}{}^{cd}_{\rm qu}\Big\}\bigg) \\
&\sim -\frac{1}{2\beta\hbar}\partial_a(H_{bcd}(x_0))\,\zeta^{cd}\int_{-1}^0 \mathrm{d}\tau\,\Big(\overline{\xi}{}^a_{\rm qu}\xi^b_{\rm qu}+\xi^a_{\rm qu}\overline{\xi}{}^b_{\rm qu}\Big) \\
&\quad -\frac{1}{2\beta\hbar}\partial_a(H_{bcd}(x_0))\,\zeta^{ab}\int_{-1}^0 \mathrm{d}\tau\,\Big(\xi^{cd}_{\rm qu}+\overline{\xi}{}^{cd}_{\rm qu}\Big)\,.
\end{aligned} \tag{7.21}$$

Notice that the first line in (7.21) does not contribute to the amplitudes: each term is contracted to the terms in the same line, which yields zero amplitudes because the background fermionic fields are anti-symmetric in the amplitudes in such a way as $\zeta^{abc}\zeta^{\rm def} = -\zeta^{\rm def}\zeta^{abc}$. Now, we combine (7.19b) and (7.21) to yield

$$\begin{aligned}
\text{(7.19b)+(7.21)} &= -\frac{1}{4\beta\hbar}R_{cdab}(\omega_+(x_0))\,\zeta^{ab}\int_{-1}^0 \mathrm{d}\tau\,\xi^{cd}_{\rm qu} - \frac{1}{4\beta\hbar}R_{cdab}(\omega_+(x_0))\,\zeta^{ab}\int_{-1}^0 \mathrm{d}\tau\,\overline{\xi}{}^{cd}_{\rm qu} \\
&\quad + \frac{1}{2\beta\hbar}\Big(R_{cdab}(\omega(x_0)) - \partial_a(H_{bcd}(x_0)) + \partial_b(H_{acd}(x_0))\Big)\zeta^{cd}\int_{-1}^0 \mathrm{d}\tau\,\xi^a_{\rm qu}\overline{\xi}{}^b_{\rm qu} \\
&= -\frac{1}{4\beta\hbar}R_{cdab}(\omega_+(x_0))\zeta^{ab}\int_{-1}^0\mathrm{d}\tau\Big\{\xi^{cd}_{\rm qu}+\overline{\xi}{}^{cd}_{\rm qu}\Big\}+\frac{1}{2\beta\hbar}R_{cdab}(\omega_-(x_0))\zeta^{cd}\int_{-1}^0\mathrm{d}\tau\,\xi^a_{\rm qu}\overline{\xi}{}^b_{\rm qu} \\
&= \frac{1}{2\beta\hbar}R_{abcd}(\omega_+(x_0))\,\zeta^{ab}\bigg(-\frac{1}{2}\int_{-1}^0\mathrm{d}\tau\,\xi^{cd}_{\rm qu} - \frac{1}{2}\int_{-1}^0\mathrm{d}\tau\,\overline{\xi}{}^{cd}_{\rm qu} + \int_{-1}^0\mathrm{d}\tau\,\xi^c_{\rm qu}\overline{\xi}{}^d_{\rm qu}\bigg)\,.
\end{aligned} \tag{7.22}$$

Then, we rewrite the action (7.7b) by summarizing (7.19a), (7.21) and (7.22) in the following form:

$$\begin{aligned}
-\frac{1}{\hbar}S_H^{(\rm int)} &= -\frac{1}{2\beta\hbar}R_{mnab}(\omega_+(x_0))\,\zeta^{ab}\int_{-1}^0\mathrm{d}\tau\,q^m\dot{q}^n \\
&\quad + \frac{1}{2\beta\hbar}R_{abcd}(\omega_+(x_0))\,\zeta^{ab}\bigg(-\frac{1}{2}\int_{-1}^0\mathrm{d}\tau\xi^c_{\rm qu}\xi^d_{\rm qu} - \frac{1}{2}\int_{-1}^0\mathrm{d}\tau\overline{\xi}{}^c_{\rm qu}\overline{\xi}{}^d_{\rm qu} + \int_{-1}^0\mathrm{d}\tau\xi^c_{\rm qu}\overline{\xi}{}^d_{\rm qu}\bigg)\,.
\end{aligned} \tag{7.23}$$

This form is exactly same as (7.9b). Then the path integral of (7.23) is also given in the same as (7.16):

$$\sigma = \int \mathrm{d}^D x_0 \sqrt{g(x_0)} \prod_{a=1}^{D} \mathrm{d}\zeta^a \, \exp\left[\frac{1}{2}\mathrm{tr}\log\left(\frac{-iR^{(+)}/2\pi}{\tanh(-iR^{(+)}/2\pi)}\right)\right] , \tag{7.24a}$$

$$R^{(+)}_{mn} = \frac{1}{2} R_{mnab}(\omega_+(x_0))\, \zeta^a \zeta^b , \tag{7.24b}$$

or, if we integrate out the fermionic fields and using the following formula (in the same way as (6.12))

$$\int \prod_{a=1}^{D} \mathrm{d}\zeta^a \, \zeta^{a_1 \cdots a_D} = \int \mathrm{d}\zeta^1 \mathrm{d}\zeta^2 \cdots \mathrm{d}\zeta^D \, \zeta^1 \zeta^2 \cdots \zeta^D \cdot \mathcal{E}^{a_1 a_2 \cdots a_D} = (-1)^{D/2} \mathcal{E}^{a_1 a_2 \cdots a_D} , \tag{7.25}$$

we simplify (7.24) and obtain

$$\sigma = \int \exp\left[\frac{1}{2}\mathrm{tr}\log\left(\frac{iR^{(+)}/2\pi}{\tanh(iR^{(+)}/2\pi)}\right)\right] , \tag{7.26a}$$

$$\mathrm{tr}(R^k_{(+)}) = R^{(+)}_{m_1 n_1} R^{(+)}_{m_2 n_2} \cdots R^{(+)}_{m_k n_k} \, g^{n_1 m_2} g^{n_2 m_3} \cdots g^{n_k m_1} , \tag{7.26b}$$

$$R^{(+)}_{mn} = \frac{1}{2} R_{mnab}(\omega_+(x_0))\, e^a \wedge e^b . \tag{7.26c}$$

8. Summary and discussions

In this paper we have studied various topological invariants on the torsional manifold in the framework of supersymmetric quantum mechanical path integral formalism. First we constructed the $\mathcal{N}=1$ supersymmetric quantum mechanics (2.6) whose target space corresponds to the torsional manifold. We extended this to the $\mathcal{N}=2$ quantum mechanics (2.16) with introducing a closed condition of the torsion. Next we described the transition elements which appear in the calculation of the Witten index. Following the work [26], we rewrote the transition elements from the Hamiltonian formalism to the Lagrangian formalism (3.20) in the $\mathcal{N}=2$ case, and (3.22) in the $\mathcal{N}=1$ case. Since we have already known these topological invariants on the Riemannian manifold in the framework of the quantum mechanical path integral, we applied the same formalism to the analyses of the Witten indices. Then we realized the formulation of the Dirac index on the torsional manifold (5.17) which have already been investigated by Mavromatos [20], Yajima [21], Peeters and Waldron [22], and so forth. The point is that we should carefully use the Riemann normal coordinate frame on the spin connection (and the affine connection) equipped with torsion. We also analyzed the Euler characteristic (6.18) and the Hirzebruch signature (7.26) on the torsional manifold. These modified values should also be topological invariants because we started from the well-defined supersymmetric algebras (2.1) in the $\mathcal{N}=1$ case and (2.15) in the $\mathcal{N}=2$ case, respectively. In these systems we can define the bosonic and fermionic states whose energy levels are degenerated. We should also find the zero energy eigenstates, which gives the Witten index as the topological value. We evaluated these Witten indices in various supersymmetric systems.

The most significant result in this paper is that the Euler characteristic (6.18) is not modified even in the presence of torsion, while the Dirac index (5.17) and the Hirzebruch signature (7.26) are. Then we conclude that if the compactified manifold has the Bismut torsion (1.2d) with the constraint (1.3d) in string theory compactification scenarios, the numbers of generation in the four-dimensional effective theory is not changed from the numbers of generation derived from the corresponding Calabi-Yau manifold without the torsion.

In this paper we imposed the closed condition $\mathrm{d}H = 0$ on the totally anti-symmetric torsion. Peeters and Waldron [22] have already investigated the Dirac index on a four-dimensional geometry with boundary in the presence of a totally anti-symmetric torsion H, and have discussed the role of $\mathrm{d}H$ in the Feynman graphs. The four-form $\mathrm{d}H$ can be described as the Nieh-Yan four-form $\mathcal{N}(e, H) = \mathrm{d}(e^A \wedge H_A)$, which appears in [32] and is applied to the analysis of the chiral anomaly [33], and the Dirac index [22]. To complete the analysis of the index theorems on a torsional geometry in the presence of non-vanishing $\mathrm{d}H$ is of particular importance when we study the string theory compactified on a G-structure manifold [27, 8, 24].

This four-form $\mathrm{d}H$ also appears and plays a crucial role in the anomaly cancellation mechanism in heterotic string theory (see [34, 19, 26] as instructive references). In the usual anomaly cancellation in heterotic string, the Bianchi identity of the NS-NS three-form H is given in terms of the Riemann curvature two-form and the field strength of the gauge field [35]: $\mathrm{d}H = -\alpha'[\mathrm{tr}\{R(\omega) \wedge R(\omega)\} - \mathrm{tr}(F \wedge F)]$. In the presence of non-vanishing H-flux, the spin connection ω in the Bianchi identity is modified to $\omega_{+MAB} = \omega_{MAB} + H_{MAB}$ and the Bianchi identity is rewritten such as

$$\mathrm{d}H = -\alpha' \Big[\mathrm{tr}\big\{ R(\omega_+) \wedge R(\omega_+) \big\} - \mathrm{tr}\big(F \wedge F \big) \Big] \, . \tag{8.1}$$

The modification of the Bianchi identity (8.1) was, for instance, investigated by Hull [36] in the framework of the worldsheet sigma model. Bergshoeff and de Roo applied (8.1) to the supergravity Lagrangian with higher-order α' corrections [37]. Recent papers follow this modification and analyze the structures in the effective theories from the heterotic string (see, for instance, [12, 13, 24, 38 – 41] and references therein). Since, even in the presence of the condition $\mathrm{d}H = 0$, we have completed the derivation of topological invariants which will contribute to the anomaly in string theory, we will be able to derive the above modified Bianchi identity in the flux compactification scenarios in an explicit way.

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A. Convention

We introduce vielbeins $e_M{}^A$ and their inverses $E_A{}^M$, which come from the spacetime metric g_{MN} and the metric η_{AB} on orthogonal frame via $g_{MN} = \eta_{AB}\, e_M{}^A\, e_N{}^B$ and $\eta_{AB} = g_{MN}\, E_A{}^M\, E_B{}^N$. By using these geometrical variables, let us define the covariant derivatives $D_M(\omega, \Gamma)$ in such a way as

$$\begin{aligned}
D_M(\Gamma) A_N &= \partial_M A_N - \Gamma^P{}_{NM} A_P \ , && \text{(A.1a)} \\
D_M(\Gamma) A^N &= \partial_M A^N + \Gamma^N{}_{PM} A^P \ , && \text{(A.1b)} \\
D_M(\Gamma) g_{NP} &\equiv 0 = \partial_M g_{NP} - \Gamma^Q{}_{NM} g_{QP} - \Gamma^Q{}_{PM} g_{NQ} \ , && \text{(A.1c)} \\
D_M(\Gamma) g^{NP} &\equiv 0 = \partial_M g^{NP} + \Gamma^N{}_{QM} g^{QP} + \Gamma^P{}_{QM} g^{NQ} \ , && \text{(A.1d)} \\
D_M(\omega, \Gamma) e_N{}^A &\equiv 0 = \partial_M e_N{}^A + \omega_M{}^A{}_B\, e_N{}^B - \Gamma^P{}_{NM}\, e_P{}^A \ , && \text{(A.1e)} \\
D_M(\omega, \Gamma) E_A{}^N &\equiv 0 = \partial_M E_A{}^N - E_B{}^N\, \omega_M{}^B{}_A + \Gamma^N{}_{PM}\, E_A{}^P \ , && \text{(A.1f)} \\
[D_M(\Gamma), D_N(\Gamma)] A_Q &= -R^P{}_{QMN}(\Gamma) A_P + 2 T^P{}_{MN}\, D_Q(\Gamma) A_P \ , && \text{(A.1g)} \\
R^P{}_{QMN}(\Gamma) &= \partial_M \Gamma^P{}_{QN} - \partial_N \Gamma^P{}_{QM} + \Gamma^P{}_{RM} \Gamma^R{}_{QN} - \Gamma^P{}_{RN} \Gamma^R{}_{QM} \ . && \text{(A.1h)}
\end{aligned}$$

Note that A_M in the above equations are vector. $\Gamma^P{}_{MN}$ is the affine connection whose two lower indices are not symmetric in general case. The anti-symmetric part of the affine connection $\Gamma^P{}_{[MN]}$ is defined as a torsion $T^P{}_{MN}$, while the symmetric part $\Gamma^P{}_{(MN)}$ is given in terms of the Levi-Civita connection Γ^P_{0MN} and torsion terms in the following way:

$$\begin{aligned}
\Gamma^P{}_{MN} &= \Gamma^P{}_{(MN)} + \Gamma^P{}_{[MN]} \ , && \text{(A.2a)} \\
\Gamma^P{}_{[NM]} &= T^P{}_{NM} \ , \quad \Gamma^P{}_{(MN)} = \Gamma^P_{0MN} - T_M{}^P{}_N - T_N{}^P{}_M \ , && \text{(A.2b)} \\
\Gamma^P_{0MN} &= \frac{1}{2} g^{PQ} \big(\partial_M g_{QN} + \partial_N g_{MQ} - \partial_Q g_{MN} \big) \ . && \text{(A.2c)}
\end{aligned}$$

Then the affine connection is also given in terms of the Levi-Civita connection and the other:

$$\Gamma^P{}_{MN} = \Gamma^P_{0MN} + K^P{}_{MN} \ , \quad K^P{}_{MN} \equiv T^P{}_{MN} - T_M{}^P{}_N - T_N{}^P{}_M \ . \tag{A.3}$$

The tensor $K^P{}_{MN}$ is called the contorsion.

We also introduce the covariant derivative induced by the local Lorentz transformation acting on a generic field ϕ^i as

$$D_M(\omega) \phi^i = \Big\{ \delta^i_j\, \partial_M - \frac{i}{2} \omega_M{}^{AB} (\Sigma_{AB})^i{}_j \Big\} \phi^j \ , \tag{A.4}$$

where Σ_{AB} is the Lorentz generator whose explicit form depends on the representation of the field ϕ^i. The curvature tensor associated with this covariant derivative is given in terms of the spin connection

$$\begin{aligned}
[D_M(\omega), D_N(\omega)] \phi \ &= \ -\frac{i}{2} R^{AB}{}_{MN}(\omega)\, \Sigma_{AB} \phi \ , && \text{(A.5a)} \\
R^{AB}{}_{MN}(\omega) \ &= \ \partial_M \omega_N{}^{AB} - \partial_N \omega_M{}^{AB} + \omega_M{}^A{}_C\, \omega_N{}^{CB} - \omega_N{}^A{}_C\, \omega_M{}^{CB} \ . && \text{(A.5b)}
\end{aligned}$$

We also describe the first and second Bianchi identity on Riemann tensor:

$$\begin{aligned}
&\text{1st:} \quad 0 = R^M{}_{NPQ}(\Gamma_0) + R^M{}_{PQN}(\Gamma_0) + R^M{}_{QNP}(\Gamma_0) \ , && \text{(A.6a)} \\
&\text{2nd:} \quad 0 = \nabla_M R^N{}_{PQR}(\Gamma_0) + \nabla_Q R^N{}_{PRM}(\Gamma_0) + \nabla_R R^N{}_{PMQ}(\Gamma_0) \ . && \text{(A.6b)}
\end{aligned}$$

B. Formulae

In the formulation of discretized and continuum path integral in quantum mechanics, we define a number of functions without ambiguities [26]. Here let us summarize functions which appear in propagators and their derivatives in the quantum mechanics.

$$\Delta(\sigma,\tau) = \sigma(\tau+1)\theta(\sigma-\tau) + \tau(\sigma+1)\theta(\tau-\sigma) = \Delta(\tau,\sigma) \,, \tag{B.1a}$$

$$\theta(\sigma-\tau)\big|_{\tau=\sigma} = \frac{1}{2} \,, \qquad \theta(\tau-\sigma) = -\theta(\sigma-\tau) + 1 \,, \tag{B.1b}$$

$$\partial_\sigma \theta(\sigma-\tau) = \delta(\sigma-\tau) \,, \quad \partial_\sigma^2 \Delta(\sigma,\tau) = \delta(\sigma-\tau) \,, \tag{B.1c}$$

$$\int_{-1}^0 \mathrm{d}\sigma \int_{-1}^0 \mathrm{d}\tau \, \Delta(\sigma,\tau) = -\frac{1}{12} \,, \quad \int_{-1}^0 \mathrm{d}\sigma \int_{-1}^0 \mathrm{d}\tau \, \delta(\sigma-\tau)\, \theta(\sigma-\tau)\, \theta(\tau-\sigma) = \frac{1}{4}. \tag{B.1d}$$

Notice that $\delta(\sigma-\tau)$ should be regarded as the "Kronecker delta" instead of the delta function because this function appears in the discretized form of the path integral and we should take the continuum limit carefully.

By using the above basic functions, we should compute various kinds of integral when we analyze loop diagrams in the path integral formalism. In this paper we mainly use a set of useful formulae which appear in the derivation of invariant polynomials such as the Dirac genus, the Chern characters, the Hirzebruch signature, and so forth. Here we only list the formula for these invariant polynomials. When we derive the Dirac genus, we use the integral I_k defined as

$$I_k \equiv \int_{-1}^0 \mathrm{d}\tau_1 \cdots \int_{-1}^0 \mathrm{d}\tau_k \partial_{\tau_1}\Delta(\tau_1,\tau_2)\partial_{\tau_2}\Delta(\tau_2,\tau_3)\cdots\partial_{\tau_{k-1}}\Delta(\tau_{k-1},\tau_k)\partial_{\tau_k}\Delta(\tau_k,\tau_1), \tag{B.2a}$$

$$\partial_{\tau_i}\Delta(\tau_i,\tau_{i+1}) = \tau_i + \theta(\tau_i - \tau_{i+1}) \,, \quad \sum_{k=2}^{\infty} \frac{y^k}{k} I_k = \log \frac{y/2}{\sinh(y/2)} \,. \tag{B.2b}$$

The following two integrals play important roles in the derivations of the Chern classes and the Hirzebruch signature:

$$\int_{-1}^0 \mathrm{d}\sigma_1 \int_{-1}^0 \mathrm{d}\sigma_2 \cdots \int_{-1}^0 \mathrm{d}\sigma_k \, \theta(\sigma_1-\sigma_2)\theta(\sigma_2-\sigma_3)\cdots\theta(\sigma_{k-1}-\sigma_k)\theta(\sigma_k-\sigma_1) = 0 \,, \tag{B.3a}$$

$$\int_{-1}^0 \mathrm{d}\sigma_1 \int_{-1}^0 \mathrm{d}\sigma_2 \cdots \int_{-1}^0 \mathrm{d}\sigma_k \, \theta(\sigma_1-\sigma_2)\theta(\sigma_2-\sigma_3)\cdots\theta(\sigma_{k-1}-\sigma_k) = \frac{1}{k!} \,, \tag{B.3b}$$

for $k \geq 2$. We also use the following integral when we derive the Hirzebruch signature:

$$J_{2\ell} = \int_{-1}^0 \prod_{i=1}^{\ell} \mathrm{d}\tau_i \, \mathrm{d}\sigma_i \, \theta(\tau_1-\sigma_\ell)\theta(\tau_1-\sigma_1)\, \theta(\tau_2-\sigma_1)\theta(\tau_2-\sigma_2)\cdots\theta(\tau_\ell-\sigma_{\ell-1})\theta(\tau_\ell-\sigma_\ell) \,, \tag{B.4a}$$

$$\sum_{\ell=1}^{\infty} \frac{(-1)^{\ell+1}}{\ell} y^{2\ell} J_{2\ell} = \log\big(\cosh y\big) \,. \tag{B.4b}$$

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